Hydroinformatik II: Partielle Differentialgleichungen (PDEs)

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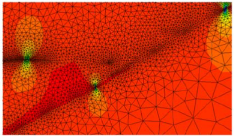
Vorlesungsplan Hydroinformatik II SoSe 2015

#	Datum	Thema
01	17.04.2015	Einführung, Grundlagen: Kontinuumsmechanik
02	24.04.2015	Grundlagen: Kontinuumsmechanik/Hydromechanik
-	01.05.2015	Maifeiertag
03	08.05.2015	HW: Einführung in Qt (Installation)
04	15.05.2015	Grundlagen: Partielle Differentialgleichungen / TEX
05	22.05.2015	Grundlagen: Numerische Methoden
-	29.05.2015	Pfingsten
06	05.06.2016	Numerik: Finite Differenzen Methode
07	12.06.2015	Grundlagen: Diffusionsprozesse
08	19.06.2015	Numerik: Übung explizite FDM
09	26.06.2015	Numerik: Implizite FDM
10	03.07.2015	Gerinnehydraulik: Theorie - Grundlagen
11	10.07.2015	Gerinnehydraulik: Saint-Venant Gleichung ($\Longrightarrow$ HSA)
12	17.07.2015	Gerinnehydraulik: Programmierung, Übung
13	17.07.2015	Kurs-Zusammenfassung und Abschluss

Konzept

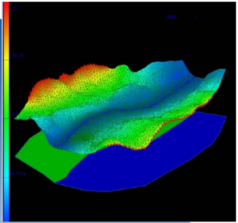
$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Basics

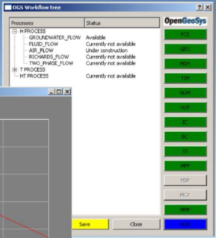
Mechanik

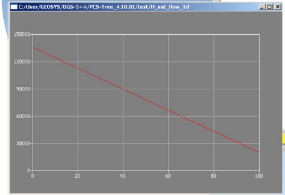
Anwendung



Numerische

Methoden





Programmierung

Visual C++

Prozessverständnis

Inhalte

- Link zur letzten Vorlesung: Dimensionsanalyse

- Konzept
- Partielle Differentialgleichungen
- Klassifikation
- Beispiele
- Anfangs- und Randbedingungen

- TEXfür den Beleg

Dimensionsanalyse

Navier-Stokes-Gleichung

Mathematical Classification (1.5)

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, ..., \frac{\partial^n \psi}{\partial x_i^n}) = 0 \quad , \quad i = 3 \tag{1}$$

where L is s differential operator. Second-order PDE with two independent variables are given by

$$A\frac{\partial^2 \psi}{\partial x^2} + B\frac{\partial^2 \psi}{\partial x \partial y} + C\frac{\partial^2 \psi}{\partial y^2} + D\frac{\partial \psi}{\partial x} + E\frac{\partial \psi}{\partial y} + F\psi + G = 0 \tag{2}$$

Second-order PDEs with more independent variables can be classified by examination of the eigenvalues of the matrix a_{ij}.

$$\sum_i \sum_j a_{ij} \frac{\partial \psi^2}{\partial x_i \partial x_j} + G = 0 \tag{3}$$

Mathematical Classification (1.5)

PDE type	Discriminant	Eigenvalues	Canonical form	Example
Elliptic	$B^2 - 4AC < 0$ complex characteristics	$\forall\lambda > 0$ equal signs	$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0$	Laplace equation
Parabolic	$B^2 - 4AC = 0$	$\exists\lambda = 0$	$\frac{\partial^2\psi}{\partial\eta^2} = G$	Diffusion, Burgers equations
Hyperbolic	$B^2 - 4AC > 0$ real characteristics	$\exists\lambda < 0$ different signs	$\frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta^2} = 0$	Wave equation

General Balance Equation (1.1.7)

- Integral form

$$\int_\Omega \frac{d\psi}{dt} d\Omega = \int_\Omega \frac{\partial\psi}{\partial t} d\Omega + \int_\Omega \nabla \cdot (\mathbf{v}\psi) d\Omega - \int_\Omega \nabla \cdot (\mathbf{D}^\psi \nabla\psi) d\Omega = \int_\Omega Q^\psi d\Omega \tag{4}$$

- Differential form

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla\psi) = \mathbf{Q}^\psi \tag{5}$$

PDE: Definition

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi}\nabla\psi) = \mathbf{Q}^{\psi} \tag{6}$$

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial\psi}{\partial x_i}, ..., \frac{\partial^n\psi}{\partial x_i^n}) = 0 \quad , \quad i = 3 \tag{7}$$

where L is s differential operator.

PDE: Klassifikation

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi}\nabla\psi) = \mathbf{Q}^{\psi} \quad (8)$$

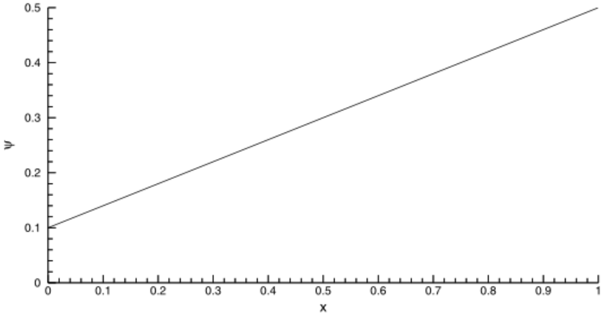
Physical problem	Math. problem	Examples
Equilibrium problems	Elliptic equations	Irrotational incompressible flow Inviscid incompressible flow Steady state heat conduction
Propagation problems (infinite propagation speed)	Parabolic equations	Unsteady viscous flow Transient heat transfer
Propagation problems (finite propagation speed)	Hyperbolic equations	Wave propagation (vibration) Inviscid supersonic flow

- Parabolisch: Diffusion, Gerinne (nichtlinear)
- Elliptisch: Grundwasser (stationär)

PDE: Elliptic Equation 1-D

$$\frac{d^2\psi}{dx^2} = 0 \tag{9}$$

$$\psi = ax + b \tag{10}$$



PDE: Elliptic Equation 2-D

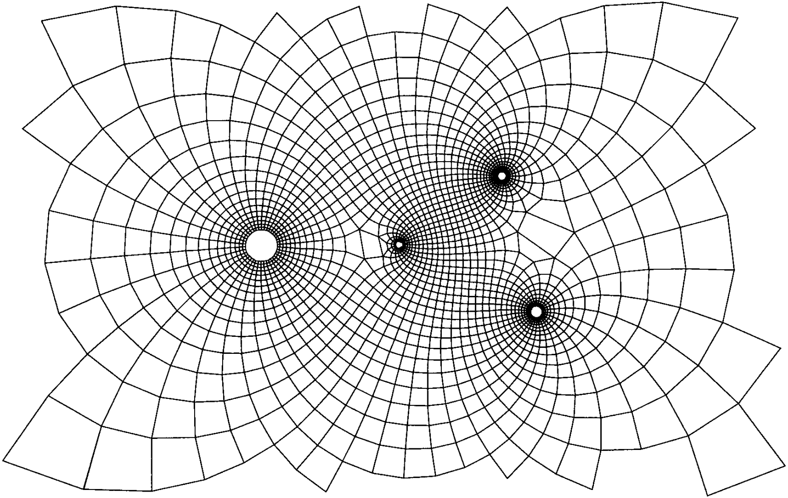
The prototype of an elliptic equation is the Laplace equation.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{11}$$

By substitution it can be easily verified that the exact solution of the Laplace equation is

$$\psi = sin(\pi x) exp(-\pi y) \tag{12}$$

PDE: Elliptic Equation 2-D



PDE: Parabolic Equation 1-D

$$\frac{\partial \psi}{\partial t} = \alpha \frac{\partial^2 \psi}{\partial x^2} \quad (13)$$

Multiple solutions:

$$\psi(t, x) = sin(\sqrt{\pi\alpha}x)exp(-\pi t) \quad (14)$$

$$\psi(t, x) = sin(\pi x)exp(-\pi^2 t) \quad (15)$$

PDE: Parabolic Equation 1-D

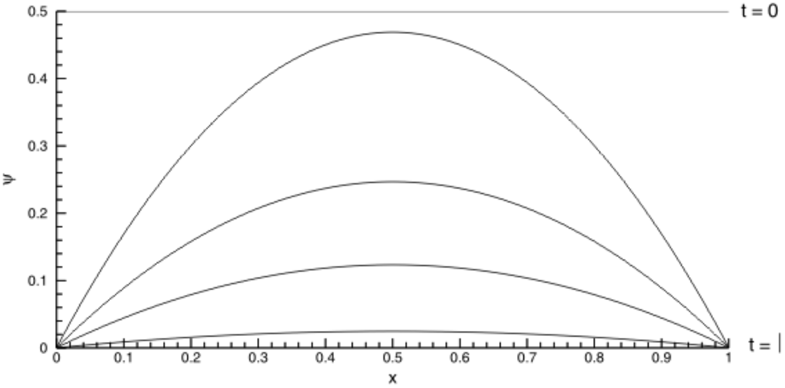


Abbildung: Solution of a parabolic equation

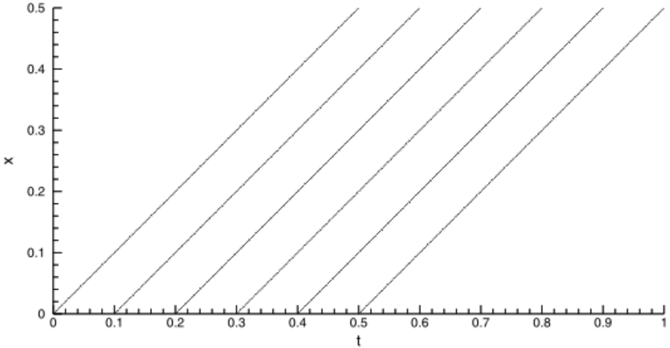
PDE: Hyperbolic Equation 1-D

$$\frac{\partial^2\psi}{\partial^2 t} - c^2\frac{\partial^2\psi}{\partial^2 x} = 0 \tag{16}$$

$$\psi(t,x) = a\cos(\frac{\pi ct}{L})\sin(\frac{\pi x}{L}) \tag{17}$$

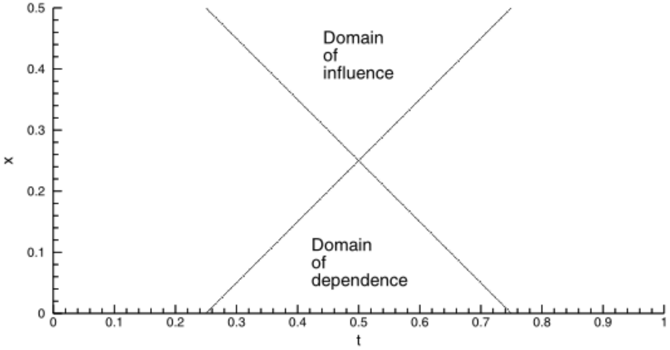
PDE: Hyperbolic Equation 1-D - Characteristics

$$A\frac{\partial\psi}{\partial t} - B\frac{\partial\psi}{\partial x} = 0 \tag{18}$$



PDE: Hyperbolic Equation 1-D - Characteristics

$$A\frac{\partial\psi}{\partial t} - B\frac{\partial\psi}{\partial x} = 0 \tag{19}$$



PDE: Equation Types

The following table gives typical examples of balance equations for the denoted quantities and their PDE types.

Physical meaning	Equation structure	Examples
Continuity	$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$	Laplace equation
Mass/energy	$\frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} - \alpha\frac{\partial^2 \psi}{\partial x^2} = 0$	Fokker-Planck equation
Momentum	$\frac{\partial \psi}{\partial t} + \psi\frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x}[\alpha(\psi)\frac{\partial \psi}{\partial x}] = 0$	Navier-Stokes equation

Boundary Conditions I

The following table gives an overview on common boundary condition types and its mathematical representation.

Tabelle: Boundary conditions types

Type of BC	Mathematical Meaning	Physical Meaning
Dirichlet type	ψ	prescribed value potential surface
Neumann type	$\nabla\psi$	prescribed flux stream surface
Cauchy type	$\psi + A\nabla\psi$	resistance between potential and stream surface

Boundary Conditions II

To describe conditions at boundaries we can use flux expressions of conservation quantities.

Tabelle: Fluxes through surface boundaries

Quantity	Flux term
Mass	$\rho \mathbf{v}$
Momentum	$\rho \mathbf{v}\mathbf{v} - \sigma$
Energy	$\rho e \mathbf{v} - \lambda \nabla \mathbf{T}$

TEX1#3

```
%==================================================================
\documentclass[twoside]{report} % double side
%%\documentclass[]{report} % single side
%------------------------------------------------------------------
% Pakete
\usepackage[dvips]{graphicx}
\usepackage{epsfig}
\usepackage{german}        % Verwenden der deutschen Trennmuster
\usepackage[ansi]{umlaute} % Unterstuetzen von deutschen Umlauten
\hyphenation{me-cha-nik}   % Trennmuster fuer Ausnahmefaelle
%------------------------------------------------------------------
% Formatierung
\setlength{\parindent}{0pt} % Absaetze nicht einruecken
\setlength{\parskip}{5pt plus 2pt minus 1pt}
\setcounter{secnumdepth}{5} %
\usepackage{fheading}        % Seitenkopf gestalten
\pagestyle{fancy}
\lhead[\fancyplain{}{\footnotesize\textsf\thepage}]%
      {\fancyplain{}{\footnotesize\textsf\rightmark}}
\rhead[\fancyplain{}{\footnotesize\textsf\leftmark}]%
      {\fancyplain{}{\footnotesize\textsf\thepage}}
\cfoot{}
...
```

TEX2#3

```
%------------------------------------------------------------
% Makros
\def \UVec {\mathbf x}
\def \AMat {\mathbf A}
\def \RHS {\mathbf b}
\def \Jacobian {\mathbf J}
\def \E {$\Rightarrow$}
%
\newcommand{\red}[1]{{\color{red}#1}}
%
\include{makros/rfsdef}
\include{makros/rfddef}
\include{makros/stnddef}
% --- Fonts
\include{fonts}
```

TEX3#3

```
%==========================================================
\begin{document}
%----------------------------------------------------------
\thispagestyle{empty}
\include{titel}
%----------------------------------------------------------
\include{exam2012-13WS_Beleg}
%----------------------------------------------------------
% Citations
\nocite{Kol:2002}
\bibliographystyle{plain} % unsrt}
\bibliography{software}
%----------------------------------------------------------
% Inhaltsverzeichnis
\tableofcontents
%----------------------------------------------------------
\end{document}
%==========================================================
```