# Hydroinformatik II - SoSe 2024

# HyBHW-S2-01-V05: Partielle Differentialgleichungen

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Dresden, 28.06.2024

# Zeitplan: Hydroinformatik II - SoSe 2024

| Datum | HI | II | Thema | Typ |
|---|---|---|---|---|
| 14.06.2024 | 14 | 2-01 | Einführung in die Lehrveranstaltung - Teil 2 | L |
| 14.06.2024 | 15 | 2-02 | Werkzeuge \| Tools | L |
| 14.06.2024 | 16 | 2-03 | Grundlagen: Kontinuumsmechanik | L |
| 21.06.2024 | 17 | 2-04 | Grundlagen: Hydromechanik | L |
| 21.06.2024 | 18 | 2-05 | Grundlagen: Partielle Partialgleichungen | L |
| 21.06.2024 | 19 | 2-06 | Übung: Analytische Lösungen | E |
| 28.06.2024* | 20 | 2-07 | Grundlagen: Näherungsverfahren | L |
| 28.06.2024* | 21 | 2-08 | Übung: Jupyter Diffusionsprozess | E |
| 02.07.2024* | 22 | 2-09 | Numerik: Finite-Differenzen-Methode (explizit) | L |
| 02.07.2024* | 23 | 2-10 | Numerik: Finite-Differenzen-Methode (implizit) | L |
| 12.07.2024 | 24 | 2-11 | Übung: Finite-Differenzen-Methoden | E |
| 12.07.2024 | 25 | 2-12 | Grundlagen: Gerinnehydraulik | L |
| 12.07.2024 | 26 | 2-13 | Übung: Gerinnehydraulik | E |
| 19.07.2024 | 27 | 2-14 | Ausblick: Grundwassermodellierung | E |
| 19.07.2024 | 28 | 2-15 | Klausur/Beleg: Besprechung zur Vorbereitung | L |

*online Vorlesung

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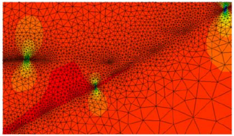
# Konzept

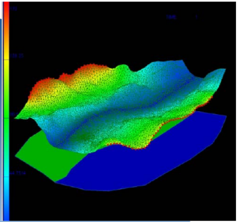
$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Basics

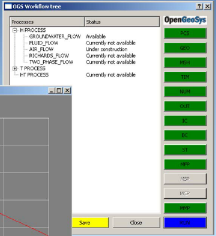
Mechanik

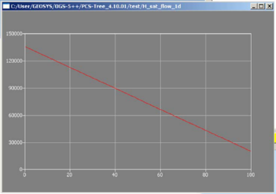
Anwendung



Numerische

Methoden




Programmierung

Visual C++

Prozessverständnis

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# Fahrplan für heute

- Konzept: Generelle Erhaltungsgesetze (Mechanik) zur Mathematik (PDEs)
- Partielle Differentialgleichungen (PDE)
- Klassifikationen
- Einfache Beispiele (Python-Übungen)
- Anfangs- und Randbedingungen

# Dimensionsanalyse

Navier-Stokes Equation

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v}} \tag{1}$$

Ableitungen:

$$\begin{aligned} &\frac{\partial}{\partial t} \\ \nabla &= \frac{\partial}{\partial x} \\ \Delta &= \frac{\partial^2}{\partial x^2} \end{aligned}$$

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# Mathematical Classification (1.5)

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, ..., \frac{\partial^n \psi}{\partial x_i^n}) = 0 \quad , \quad i = 3 \tag{2}$$

where $L$ is s differential operator. Second-order PDE with two independent variables are given by

$$A\frac{\partial^2 \psi}{\partial x^2} + B\frac{\partial^2 \psi}{\partial x \partial y} + C\frac{\partial^2 \psi}{\partial y^2} + D\frac{\partial \psi}{\partial x} + E\frac{\partial \psi}{\partial y} + F\psi + G = 0 \tag{3}$$

Second-order PDEs with more independent variables can be classified by examination of the eigenvalues of the matrix $a_{ij}$.

$$\sum_i \sum_j a_{ij} \frac{\partial \psi^2}{\partial x_i \partial x_j} + G = 0 \quad , \quad a_{ii} = \lambda_i \quad \text{Eigenvalues} \tag{4}$$

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# Mathematical Classification (1.5)

| PDE type | Discriminant | Eigenvalues | Canonical form | Example |
|---|---|---|---|---|
| Elliptic | $B^2 - 4AC < 0$<br>complex characteristics | $\forall\lambda > 0$<br>equal signs | $\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0$ | Laplace equation |
| Parabolic | $B^2 - 4AC = 0$ | $\exists\lambda = 0$ | $\frac{\partial^2\psi}{\partial\eta^2} = G$ | Diffusion, Burgers equations |
| Hyperbolic | $B^2 - 4AC > 0$<br>real characteristics | $\exists\lambda < 0$<br>different signs | $\frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta^2} = 0$ | Wave equation |

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# General Balance Equation (1.1.7)

- Integral form

$$\int_\Omega \frac{d\psi}{dt} d\Omega = \int_\Omega \frac{\partial\psi}{\partial t} d\Omega + \int_\Omega \nabla \cdot (\mathbf{v}\psi) d\Omega - \int_\Omega \nabla \cdot (\mathbf{D}^\psi \nabla\psi) d\Omega = \int_\Omega Q^\psi d\Omega \tag{5}$$

- Differential form

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla\psi) = \mathbf{Q}^\psi \tag{6}$$

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# PDE: Definition

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi}\nabla\psi) = Q^{\psi} \qquad (7)$$

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A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial\psi}{\partial x_i}, ..., \frac{\partial^n\psi}{\partial x_i^n}) = 0 \qquad , \qquad i = 3 \qquad (8)$$

where $L$ is s differential operator.

# PDE: Klassifikation

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = Q^{\psi} \qquad (9)$$

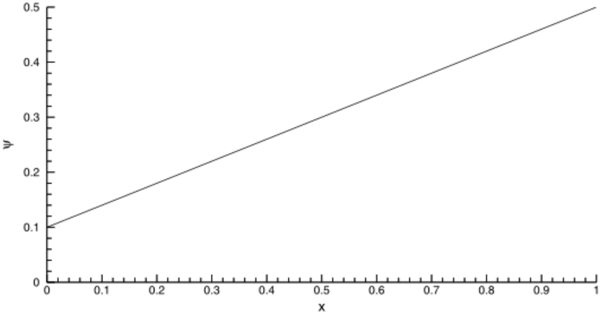
| Physical problem | Math. problem | Examples |
|---|---|---|
| Equilibrium problems | Elliptic equations | Irrotational incompressible flow<br>Inviscid incompressible flow<br>Steady state heat conduction |
| Propagation problems (infinite propagation speed) | Parabolic equations | Unsteady viscous flow<br>Transient heat transfer |
| Propagation problems (finite propagation speed) | Hyperbolic equations | Wave propagation (vibration)<br>Inviscid supersonic flow |

- Parabolisch: Diffusion, Gerinne (nichtlinear)
- Elliptisch: Grundwasser (stationär)

# PDE: Elliptic Equation 1-D

$$\frac{d^2\psi}{dx^2} = 0 \tag{10}$$

$$\psi = ax + b \tag{11}$$



# PDE: Elliptic Equation 2-D

The prototype of an elliptic equation is the Laplace equation.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{12}$$

By substitution it can be easily verified that the exact solution of the Laplace equation is

$$\psi = sin(\pi x) exp(-\pi y) \tag{13}$$

$$\frac{\partial \psi}{\partial x} = \pi cos(\pi x) exp(-\pi y), \frac{\partial \psi}{\partial y} = ..., \frac{\partial^2 \psi}{\partial x^2} = ..., \frac{\partial^2 \psi}{\partial y^2} = ... \tag{14}$$

$\Rightarrow$ Hausaufgabe (am Ende)

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# PDE: Elliptic Equation 2-D

The prototype of an elliptic equation is the Laplace equation.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{15}$$

$$\psi(x,y) = \int \left( \frac{x-L}{(x-L)^2+y^2} - \frac{x+L}{(x+L)^2+y^2} \right) dx\, dy \tag{16}$$

$$v_x = \frac{\partial \psi}{\partial x} = \frac{x-L}{(x-L)^2+y^2} - \frac{x+L}{(x+L)^2+y^2} \tag{17}$$

$$v_y = \frac{\partial \psi}{\partial y} = \frac{y}{(x-L)^2+y^2} - \frac{y}{(x+L)^2+y^2} \tag{18}$$

$\Rightarrow$ Randbedingungen sind wichtig

# Divergenzfreie Strömung

## Übung: EX04-divergenzfreie-stroemung.py

```python
import numpy as np
import matplotlib.pyplot as plt
# set up a normalized grid:
dim= 20
xarray= np.arange(-dim,dim)
yarray= np.arange(-dim,dim)
# (fluid) flow from a source at L to a sink at -L:
L = dim/2
x,y = np.meshgrid(xarray,yarray)
vx = (x-L)/((x-L)**2+y**2) - (x+L)/((x+L)**2 +y**2)
vy = y/((x-L)**2+y**2) - y/((x+L)**2 +y**2)
# plot the flow lines:
plt.figure()
plt.quiver(x,y, vx, vy, pivot='mid')
plt.xlabel("$x$-axis")
plt.ylabel("$y$-axis")
plt.axis('equal')
plt.show()
```

**Listing:** Python code for divergence-free flow (div v equals 0)

Source: https://auckland.figshare.com/articles/dataset/Chapter_6_Divergence_of_a_vector_field/5732421

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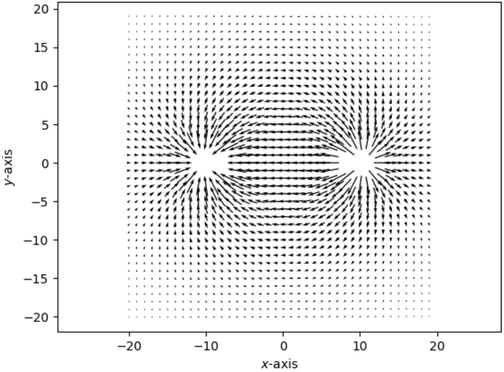
# Divergenzfreie Strömung

## Übung: EX04-divergenzfreie-stroemung.py




https://github.com/OlafKolditz/Hydroinformatik-II/blob/master/EX04-divergenzfreie-stroemung.py

Source: https://auckland.figshare.com/articles/dataset/Chapter_6_Divergence_of_a_vector_field/5732421

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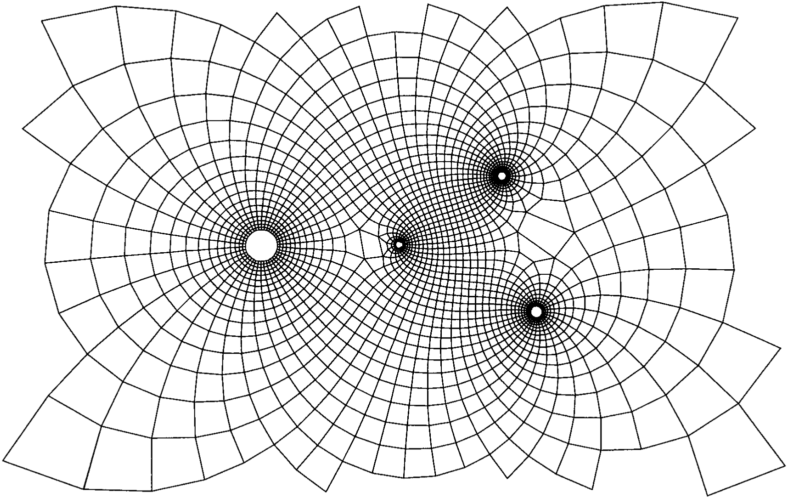
# PDE: Elliptic Equation 2-D



$$\frac{\partial \psi}{\partial t} = \alpha \frac{\partial^2 \psi}{\partial x^2} \tag{19}$$

Multiple solutions:

$$\psi(t, x) = sin(\sqrt{\pi \alpha} x) exp(-\pi t) \quad ??? \tag{20}$$

$$\psi(t, x) = sin(\frac{\pi}{\sqrt{\alpha}} x) exp(-\pi^2 t) \tag{21}$$

$$\psi(t, x) = sin(\pi x) exp(-\alpha \pi^2 t) \tag{22}$$

# PDE: Parabolic Equation 1-D

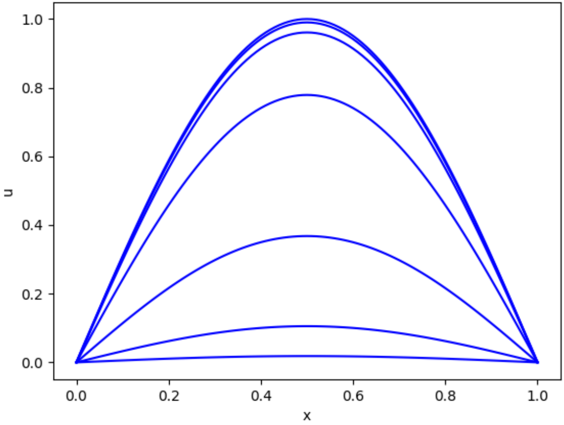



**Fig.:** Solution of a parabolic equation

```
https://github.com/OlafKolditz/Hydroinformatik-II/blob/master/EX06-parabolische-gleichung-1D.py
```

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# PDE: Hyperbolic Equation 1-D

$$\frac{\partial \psi}{\partial t} - v_x \frac{\partial \psi}{\partial x} = 0 \tag{23}$$

$\Rightarrow$ Übung

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# PDE: Hyperbolic Equation 1-D

$$\frac{\partial^2 \psi}{\partial^2 t} - c^2 \frac{\partial^2 \psi}{\partial^2 x} = 0 \tag{24}$$

$$\psi(t, x) = a \cos(\frac{\pi c t}{L}) \sin(\frac{\pi x}{L}) \tag{25}$$

$$A\frac{\partial \psi}{\partial t} - B\frac{\partial \psi}{\partial x} = 0 \tag{26}$$

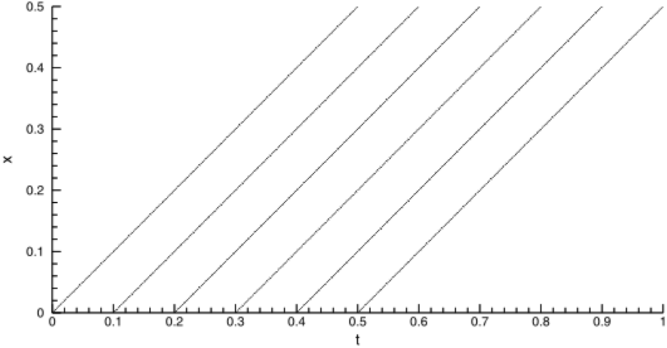


**Fig.:** Characteristics of a hyperbolic equation

# PDE: Hyperbolic Equation 1-D - Characteristics

$$A\frac{\partial\psi}{\partial t} - B\frac{\partial\psi}{\partial x} = 0 \tag{27}$$

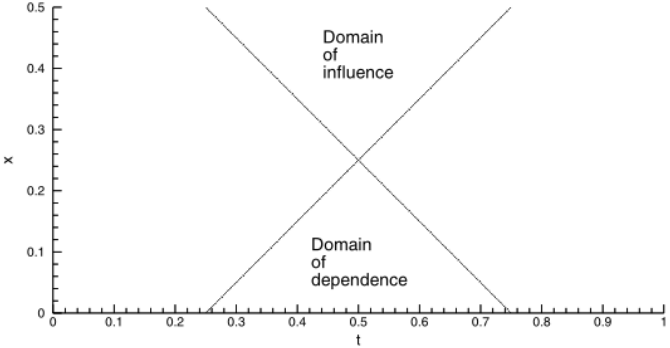


**Fig.:** Domains of a hyperbolic equation

# PDE: Hyperbolic-Parabolic Equation 1-D

Transportgleichung (advection-diffusion equation ADE)

$$\frac{\partial \psi}{\partial t} - v_x \frac{\partial \psi}{\partial x} + D_{xx} \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{28}$$

$\Rightarrow$ Übung

# PDE: Hyperbolic-Parabolic Equation 1-D

Transportgleichung (advection-diffusion equation ADE)

$$\frac{\partial \psi}{\partial t} - v_x \frac{\partial \psi}{\partial x} + D_{xx} \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{28}$$

$\Rightarrow$ Übung

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = \mathbf{Q}^{\psi} \tag{29}$$

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# PDE: Equation Types

The following table gives typical examples of balance equations for the denoted quantities and their PDE types.

| Physics | Equation structure | Examples |
|---|---|---|
| Continuity | $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$ | Laplace equation |
| Mass/energy | $\frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} - \alpha \frac{\partial^2 \psi}{\partial x^2} = 0$ | Fokker-Planck equation |
| Momentum | $\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x}[\alpha(\psi)\frac{\partial \psi}{\partial x}] = 0$ | Navier-Stokes equation |

# Boundary Conditions I

The following table gives an overview on common boundary condition types and its mathematical representation.

**Table:** Boundary conditions types

| Type of BC | Mathematical Meaning | Physical Meaning |
|---|---|---|
| Dirichlet type | $\psi$ | prescribed value<br>potential surface |
| Neumann type | $\nabla\psi$ | prescribed flux<br>stream surface |
| Cauchy type | $\psi + A\nabla\psi$ | resistance between<br>potential and stream surface |

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# Boundary Conditions II

To describe conditions at boundaries we can use flux expressions of conservation quantities.

**Table:** Fluxes through surface boundaries

| Quantity | Flux term |
|---|---|
| Mass | $\rho \mathbf{v}$ |
| Momentum | $\rho \mathbf{v}\mathbf{v} - \sigma$ |
| Energy | $\rho e \mathbf{v} - \lambda \nabla \mathbf{T}$ |

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# Hausaufgabe 03/BHYWI04-2020 vom 15.05.2020

Partielle Differentialgleichungen

**Aufgabe**: Prüfen sie die Gültigkeit der Lösungen für die partiellen Differentialgleichungen: (14), (20), (21), (22), (25).

**Lösungsweg**: Berechnen sie hierfür die entsprechenden partiellen Ableitungen und setzen sie diese dann in die entsprechenden Gleichungen (14), (20), (21), (22), (25) ein.

$$\frac{\partial \psi}{\partial t} = \ldots \quad , \quad \frac{\partial \psi}{\partial x} = \ldots \tag{30}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \ldots \quad , \quad \frac{\partial^2 \psi}{\partial y^2} = \ldots \tag{31}$$

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# Übungen: Hydroinformatik

Hydroinformatik II - HyBHW-1-02

`https://github.com/OlafKolditz/HYDROINFORMATIK-II`

- EX01: Jupyter Notebook
- EX02: Python: matplotlib
- EX03: Kontinnumsmechanik: Skalarprodukt
- EX04: Hydromechanik: Divergenzfreie Strömung
- EX05: Analytische Lösung: Elliptische Gleichung
- EX06: Analytische Lösung: Parabolische Gleichung (Diffusion)
- EX07: Analytische Lösung: Transportgleichung (ADE)
- EX08: Finite-Differenzen-Methode (FDM) explizit
- EX09: Finite-Differenzen-Methode (FDM) implizit
- EX10: Gerinnehydraulik

```
Quellcode
...
```

**Listing:** Quellcode für Übungen (C++ und Python)

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# Hausaufgaben

# Hausaufgaben: Hydroinformatik

## Hydroinformatik II - HyBHW-1-02

1. Skalarprodukt: Schreiben sie das Skalarprodukt $\nabla \cdot \mathbf{v}$ in Komponentenschreibweise.
2. Mechanik: Was ist $\mathbf{v} \cdot \nabla\psi$?
3. Mechanik: Was ist $\Phi^\psi$?
4. Hydromechanik: Komponentenschreibweise $\nabla \cdot (\mathbf{v}\psi)$
5. Hydromechanik: Komponentenschreibweise $\nabla \cdot (\mathbf{D}^\psi \nabla\psi)$
6. Analytik: Prüfen sie die Gültigkeit einer der Lösungen für die partiellen Differentialgleichungen: (17), (19), (20), (21), (23) (siehe Vorlesung 5).
7. Analytik: Darstellung der analytischen Lösung für die 1-D parabolische Differentialgleichung (BHYWI-08-02-E-Script) (31)
8. Numerik: Darstellung der numerischen Lösung (explizite FDM) für die 1-D parabolische Differentialgleichung (EX08-fdm-explicit-python)
9. Numerik: Darstellung der numerischen Lösung (implizite FDM) für die 1-D parabolische Differentialgleichung (EX09-fdm-implicit-python)
10. Numerik: Darstellung der nichtlinearen Lösung für die Gerinnehydraulik (EX10-gerinne-python)

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# Hausaufgaben: Hydroinformatik

## Beispiel: Aufgaben 7-10

- zip-File mit Übungen (Hydroinformatik-Beleg-2023-Übungen.zip) herunterladen (Lehre-Webseite)
- Python-File editieren (Matrikel-Nummer oder Name)
- Programme zum Rechnen und Darstellen ausführen
- Ergebnis in den Beleg einfügen

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