Hydroinformatik II - SoSe 2024

HyBHW-S2-01-V04: Grundlagen der Hydromechanik

Prof. Dr.-Ing. habil. Olaf Kolditz

[1]Helmholtz Centre for Environmental Research – UFZ, Leipzig

[2]Technische Universität Dresden – TUD, Dresden

[3]Center for Advanced Water Research – CAWR

[4]TUBAF-UFZ Center for Environmental Geosciences – C-EGS, Freiberg / Leipzig

Dresden, 21.06.2024

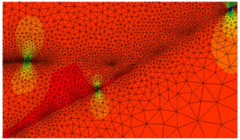
Zeitplan: Hydroinformatik II - SoSe 2024

Datum	HI	II	Thema	Typ
14.06.2024	14	2-01	Einführung in die Lehrveranstaltung - Teil 2	L
14.06.2024	15	2-02	Werkzeuge | Tools	L
14.06.2024	16	2-03	Grundlagen: Kontinuumsmechanik	L
21.06.2024	17	2-04	Grundlagen: Hydromechanik	L
21.06.2024	18	2-05	Grundlagen: Partielle Partialgleichungen	L
21.06.2024	19	2-06	Übung: Analytische Lösungen	E
28.06.2024*	20	2-07	Grundlagen: Näherungsverfahren	L
28.06.2024*	21	2-08	Übung: Jupyter Diffusionsprozess	E
02.07.2024*	22	2-09	Numerik: Finite-Differenzen-Methode (explizit)	L
02.07.2024*	23	2-10	Numerik: Finite-Differenzen-Methode (implizit)	L
12.07.2024	24	2-11	Übung: Finite-Differenzen-Methoden	E
12.07.2024	25	2-12	Grundlagen: Gerinnehydraulik	L
12.07.2024	26	2-13	Übung: Gerinnehydraulik	E
19.07.2024	27	2-14	Ausblick: Grundwassermodellierung	E
19.07.2024	28	2-15	Klausur/Beleg: Besprechung zur Vorbereitung	L

*online Vorlesung

Konzept

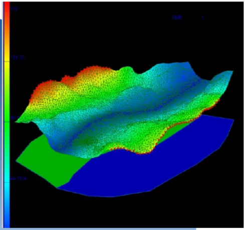
$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Basics

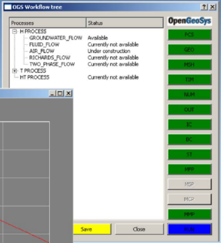
Mechanik

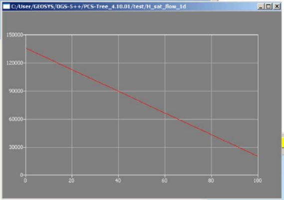
Anwendung



Numerische

Methoden





Programmierung

Visual C++

Prozessverständnis

TECHNISCHE UNIVERSITÄT DRESDEN

Fahrplan für heute

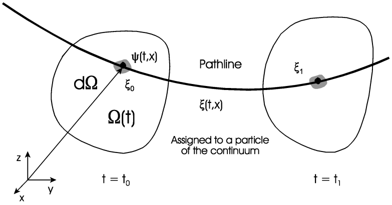
Legacy:

- git Übung
- Vorlesungsvideos
  `https://nc.ufz.de/s/fxDCcfgBdBbSfos`
  pwd: Sb29zCLBoT
- Hausaufgaben:
  HyBHW-1-02-15-Beleg

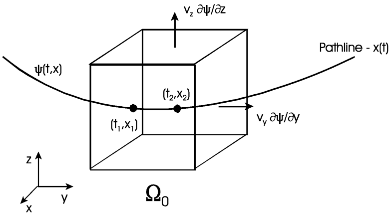
- Erhaltungsgrößen
- Massenerhaltung
- Fluidmassenerhaltung
- Diffusion
- Impulserhaltung
- Spannungen
- Fluiddruck
- Strömungsprobleme

TECHNISCHE UNIVERSITÄT DRESDEN

Lagrange



Euler



Wiederholung

General Balance Equation

$$\frac{d}{dt}\int_\Omega \psi \, d\Omega = \frac{\partial}{\partial t}\int_\Omega \psi \, d\Omega + \oint_{\partial\Omega} \mathbf{\Phi}^\psi \cdot d\mathbf{S} \qquad (1)$$

$$= \frac{\partial}{\partial t}\int_\Omega \psi \, d\Omega + \int_\Omega \nabla \cdot \mathbf{\Phi}^\psi d\Omega$$

$\lim d\Omega \to 0$

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{\Phi}^\psi \qquad (2)$$

$$= \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla \psi)$$

$$= Q^\psi$$

HA 02#2020: Komponentenschreibweise $\nabla \cdot (\mathbf{v}\psi)$, $\nabla \cdot (\mathbf{D}^\psi \nabla \psi)$

TECHNISCHE UNIVERSITÄT DRESDEN

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \tag{3}$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ, momentum density $\rho\mathbf{v}$ or energy density e.

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
Linear momentum	$\mathbf{m}$	Linear momentum density	$\rho\mathbf{v}$
Energy	E	Energy density	$e = \rho i + \frac{1}{2}\rho v^2$

(Phase) Mass Conservation

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = Q^{\psi} \tag{4}$$

The differential equation of mass conservation in divergence form becomes

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \tag{5}$$

Partial differentiation of the above equation gives

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0 \tag{6}$$

TECHNISCHE UNIVERSITÄT DRESDEN

(Phase) Mass Conservation

Using the material (or convective) derivative the mass conservation equation can be rewritten as

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{v} \tag{7}$$

Note, above convective form of mass conservation equation becomes zero only for incompressible flows, i.e.

$$\frac{\partial \rho}{\partial t} = 0 \tag{8}$$

requires divergence-free flow.

$$\nabla \cdot \mathbf{v} = 0 \tag{9}$$

From eqn. (6) results that the above expression is the continuity equation for a homogeneous fluid ($\rho = \text{const}$).

Divergenzfreie Strömung

$$\nabla \cdot \mathbf{v} = 0 \qquad (10)$$

Links:

- https://de.wikipedia.org/wiki/Stromfunktion
- https://www.ingenieurkurse.de/stroemungslehre/ebene-stroemungen/quelle-und-senke-divergenz.html

TECHNISCHE UNIVERSITÄT DRESDEN

Divergenzfreie Strömung

Übung: HyBHW-1-02-04-E1

```python
import numpy as np
import matplotlib.pyplot as plt
# set up a normalized grid:
dim= 20
xarray= np.arange(-dim,dim)
yarray= np.arange(-dim,dim)
# (fluid) flow from a source at L to a sink at -L:
L = dim/2
x,y = np.meshgrid(xarray,yarray)
vx = (x-L)/((x-L)**2+y**2) - (x+L)/((x+L)**2 +y**2)
vy = y/((x-L)**2+y**2) - y/((x+L)**2 +y**2)
# plot the flow lines:
plt.figure()
plt.quiver(x,y, vx, vy, pivot='mid')
plt.xlabel("$x$-axis")
plt.ylabel("$y$-axis")
plt.axis('equal')
plt.show()
```

Listing: Python code for divergence-free flow (div v

Source: University of Auckland

TECHNISCHE UNIVERSITÄT DRESDEN

Übungen: Hydroinformatik

Hydroinformatik II - HyBHW-1-02

`https://github.com/OlafKolditz/HYDROINFORMATIK-II`

- EX01: Jupyter Notebook
- EX02: Python: matplotlib
- EX03: Kontinnumsmechanik: Skalarprodukt
- EX04: Hydromechanik: Divergenzfreie Strömung
- EX05: Analytische Lösung: Elliptische Gleichung
- EX06: Analytische Lösung: Parabolische Gleichung (Diffusion)
- EX07: Analytische Lösung: Transportgleichung (ADE)
- EX08: Finite-Differenzen-Methode (FDM) explizit
- EX09: Finite-Differenzen-Methode (FDM) implizit
- EX10: Gerinnehydraulik

```
Quellcode
...
```

Listing: Quellcode für Übungen (C++ und Python)

Conservation Quantities (1.1.2)

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \tag{11}$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ, momentum density $\rho\mathbf{v}$ or energy density e.

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
Linear momentum	$\mathbf{m}$	Linear momentum density	$\rho\mathbf{v}$
Energy	E	Energy density	$e = \rho i + \frac{1}{2}\rho v^2$

TECHNISCHE UNIVERSITÄT DRESDEN

Momentum Conservation

$\psi = \rho \mathbf{v}$

$$\frac{\partial}{\partial t}\int_{\Omega} \rho \mathbf{v} d\Omega + \oint_{\partial\Omega} \mathbf{\Phi}^{m} \cdot d\mathbf{S} = \int_{\Omega} \rho \mathbf{f} d\Omega \tag{12}$$

Flux term: The advective momentum flux is defined as

$$\mathbf{\Phi}^{m} = (\rho \mathbf{v}) \otimes \mathbf{v} = (\rho \mathbf{v})\mathbf{v} \tag{13}$$

$$\mathbf{F} = \int_{\Omega} \rho \mathbf{f} d\Omega = \int_{\Omega} \rho(\mathbf{f}^{e} + \mathbf{f}^{i}) d\Omega = \underbrace{\int_{\Omega} \rho \mathbf{f}^{e} d\Omega}_{\text{External forces}} + \underbrace{\oint_{\partial\Omega} \boldsymbol{\sigma} : d\mathbf{S}}_{\text{Internal forces}} \tag{14}$$

Momentum Conservation

Substituting now flux and source terms of momentum we obtain

$$\frac{\partial}{\partial t}\int\limits_{\Omega}\rho\mathbf{v}d\Omega+\oint\limits_{\partial\Omega}\rho\mathbf{v}(\mathbf{v}\cdot d\mathbf{S})=\int\limits_{\Omega}\rho\mathbf{f}^{e}d\Omega+\oint\limits_{\partial\Omega}\boldsymbol{\sigma}d\mathbf{S} \tag{15}$$

Applying the Gauss-Ostrogradskian theorem to the surface integrals

$$\begin{aligned}\oint\limits_{\partial\Omega}\rho\mathbf{v}(\mathbf{v}\cdot d\mathbf{S}) &= \int\limits_{\Omega}\nabla\cdot(\rho\mathbf{v}\mathbf{v})\,d\Omega \\ \oint\limits_{\partial\Omega}\boldsymbol{\sigma}:d\mathbf{S} &= \int\limits_{\Omega}\nabla\cdot\boldsymbol{\sigma}\,d\Omega\end{aligned} \tag{16}$$

TECHNISCHE UNIVERSITÄT DRESDEN

Momentum Conservation

The differential form of the momentum conservation law is then

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \rho \mathbf{f}^e + \nabla \cdot \boldsymbol{\sigma} \tag{17}$$

The above equation is now extended by partial integration

$$\begin{aligned} &\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t} + (\rho \mathbf{v}) \cdot \nabla \mathbf{v} + \mathbf{v}\nabla \cdot (\rho \mathbf{v}) \\ = &\rho[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}] + \mathbf{v}[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})] \\ &= \rho \mathbf{f}^e + \nabla \cdot \boldsymbol{\sigma} \end{aligned} \tag{18}$$

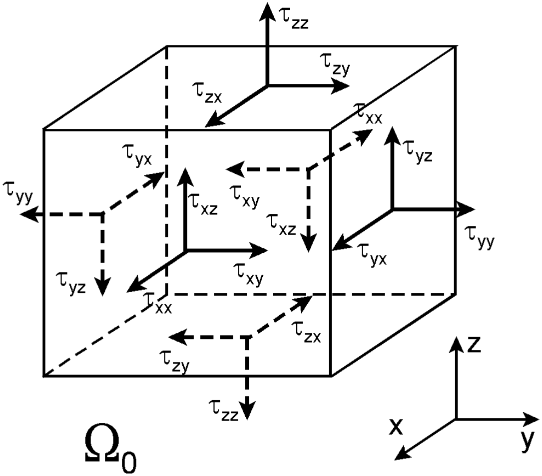
Using the mass conservation equation (5) and dividing by ρ we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} \tag{19}$$

TECHNISCHE UNIVERSITÄT DRESDEN

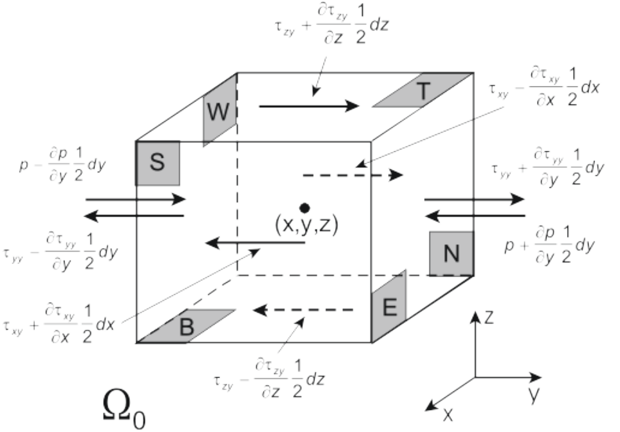
Momentum Conservation: Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \quad , \quad \boldsymbol{\tau} = \nu \nabla \mathbf{v} \tag{20}$$



Momentum Conservation: Stress Tensor

$$\boldsymbol{\tau} = \nu \nabla \mathbf{v} \tag{21}$$



Fluid Momentum Balance

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} \tag{22}$$

In index notation the above vector equation is written as

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \frac{1}{\rho}(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}) \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= \frac{1}{\rho}(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}) \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= g + \frac{1}{\rho}(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z})
\end{aligned} \tag{23}$$

with $u = v_x, v = v_y, w = v_z$ and $\mathbf{f}^e = \mathbf{g}$.

TECHNISCHE UNIVERSITÄT DRESDEN

Flow Equations - Systematic

Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \tag{24}$$

Navier-Stokes Equation

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v}} \tag{25}$$

Euler Equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p \tag{26}$$

Stokes Equation

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} \tag{27}$$

Darcy Equations

$$0 = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} \tag{28}$$

(Component) Mass Conservation

$$\psi = \rho_k = C_k \tag{29}$$

$$\frac{dC_k}{dt} = \frac{\partial C_k}{\partial t} + \nabla \cdot (\mathbf{v} C_k) - \nabla \cdot (\mathbf{D_k} \nabla C_k) = Q_k \tag{30}$$

TECHNISCHE UNIVERSITÄT DRESDEN

Übungen und Hausaufaben

- EX: Übungen
- HA: Hausaufgaben

Übungen: Hydroinformatik

Hydroinformatik II - HyBHW-1-02

https://github.com/OlafKolditz/HYDROINFORMATIK-II

- EX01: Jupyter Notebook
- EX02: Python: matplotlib
- EX03: Kontinnumsmechanik: Skalarprodukt
- EX04: Hydromechanik: Divergenzfreie Strömung
- EX05: Analytische Lösung: Elliptische Gleichung
- EX06: Analytische Lösung: Parabolische Gleichung (Diffusion)
- EX07: Analytische Lösung: Transportgleichung (ADE)
- EX08: Finite-Differenzen-Methode (FDM) explizit
- EX09: Finite-Differenzen-Methode (FDM) implizit
- EX10: Gerinnehydraulik

```
Quellcode
...
```

Listing: Quellcode für Übungen (C++ und Python)

TECHNISCHE UNIVERSITÄT DRESDEN

Hausaufgaben

Hausaufgaben: Hydroinformatik

Hydroinformatik II - HyBHW-1-02

1. Skalarprodukt: Schreiben sie das Skalarprodukt $\nabla \cdot \mathbf{v}$ in Komponentenschreibweise.
2. Mechanik: Was ist $\mathbf{v} \cdot \nabla\psi$?
3. Mechanik: Was ist Φ^{ψ}?
4. Hydromechanik: Komponentenschreibweise $\nabla \cdot (\mathbf{v}\psi)$
5. Hydromechanik: Komponentenschreibweise $\nabla \cdot (\mathbf{D}^{\psi}\nabla\psi)$
6. Analytik: Prüfen sie die Gültigkeit einer der Lösungen für die partiellen Differentialgleichungen: (17), (19), (20), (21), (23) (siehe Vorlesung 5).
7. Analytik: Darstellung der analytischen Lösung für die 1-D parabolische Differentialgleichung (BHYWI-08-02-E-Script) (26)
8. Numerik: Darstellung der numerischen Lösung (explizite FDM) für die 1-D parabolische Differentialgleichung (EX08-fdm-explicit-python)
9. Numerik: Darstellung der numerischen Lösung (implizite FDM) für die 1-D parabolische Differentialgleichung (EX09-fdm-implicit-python)
10. Numerik: Darstellung der nichtlinearen Lösung für die Gerinnehydraulik (EX10-gerinne-python)

TECHNISCHE UNIVERSITÄT DRESDEN

Hausaufgaben: Hydroinformatik

Beispiel: Aufgaben 7-10

- zip-File mit Übungen (Hydroinformatik-Beleg-2023-Übungen.zip) herunterladen (Lehre-Webseite)
- Python-File editieren (Matrikel-Nummer oder Name)
- Programme zum Rechnen und Darstellen ausführen
- Ergebnis in den Beleg einfügen