Hydroinformatik II

”Prozesssimulation und Systemanalyse”

HyBHW-1-02-04 @ 2021

Partielle Differentialgleichungen

Olaf Kolditz

*Helmholtz Centre for Environmental Research – UFZ

[1] Technische Universität Dresden – TUDD

[2] Centre for Advanced Water Research – CAWR

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Zeitplan: Hydroinformatik II

Datum	V	Thema	T
16.04.2021	01	Einführung in die Lehrveranstaltung | Tools	L
23.04.2021	02	Grundlagen: Kontinuumsmechanik	L
30.04.2021	03	Grundlagen: Hydromechanik	L
07.05.2021	04	Grundlagen: Partielle Partialgleichungen	L
14.05.2021	05a	Tools: Compiler, Python, Jupyter	E
14.05.2021	05b	Übung: Elliptische PDG	E
21.05.2021	06	Übungen: Übersicht und Werkzeuge	E
28.05.2021	–	Pfingsten	
04.06.2021	07	Grundlagen: Näherungsverfahren	L
11.06.2021	08	Numerik: Finite-Differenzen-Methode (explizit)	L
18.06.2021	09	Numerik: Finite-Differenzen-Methode (implizit)	L
25.06.2021	10	Reserve	L/E
02.07.2021	11	Übung: Diffusionsprozesse	E
09.07.2021	12	Übung: Gerinnehydraulik	E
16.07.2021	13	Übung: Grundwassermodellierung	E
23.07.2021	14	Beleg: Besprechung zur Vorbereitung	L

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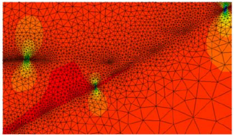
Konzept

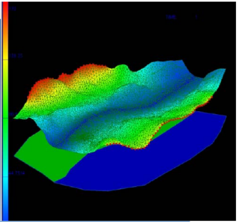
$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Basics

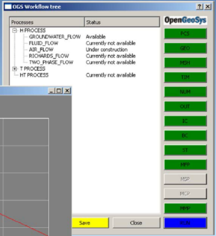
Mechanik

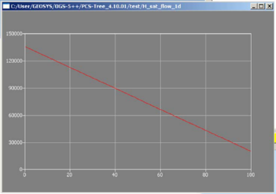
Anwendung



Numerische

Methoden





Programmierung

Visual C++

Prozessverständnis

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Inhalte

- Konzept
- Partielle Differentialgleichungen
- Klassifikation
- Beispiele
- Anfangs- und Randbedingungen

Dimensionsanalyse

Navier-Stokes-Gleichung (jetzt käme eigentlich die Tafel dran ...)

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Mathematical Classification (1.5)

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, ..., \frac{\partial^n \psi}{\partial x_i^n}) = 0 \qquad , \qquad i = 3 \tag{1}$$

where L is s differential operator. Second-order PDE with two independent variables are given by

$$A\frac{\partial^2 \psi}{\partial x^2} + B\frac{\partial^2 \psi}{\partial x \partial y} + C\frac{\partial^2 \psi}{\partial y^2} + D\frac{\partial \psi}{\partial x} + E\frac{\partial \psi}{\partial y} + F\psi + G = 0 \tag{2}$$

Second-order PDEs with more independent variables can be classified by examination of the eigenvalues of the matrix a_{ij}.

$$\sum_i \sum_j a_{ij} \frac{\partial \psi^2}{\partial x_i \partial x_j} + G = 0 \tag{3}$$

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Mathematical Classification (1.5)

PDE type	Discriminant	Eigenvalues	Canonical form	Example
Elliptic	$B^2 - 4AC < 0$ complex characteristics	$\forall\lambda > 0$ equal signs	$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0$	Laplace equation
Parabolic	$B^2 - 4AC = 0$	$\exists\lambda = 0$	$\frac{\partial^2\psi}{\partial\eta^2} = G$	Diffusion, Burgers equations
Hyperbolic	$B^2 - 4AC > 0$ real characteristics	$\exists\lambda < 0$ different signs	$\frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta^2} = 0$	Wave equation

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General Balance Equation (1.1.7)

- Integral form

$$\int_\Omega \frac{d\psi}{dt} d\Omega = \int_\Omega \frac{\partial\psi}{\partial t} d\Omega + \int_\Omega \nabla \cdot (\mathbf{v}\psi) d\Omega - \int_\Omega \nabla \cdot (\mathbf{D}^\psi \nabla\psi) d\Omega = \int_\Omega Q^\psi d\Omega \tag{4}$$

- Differential form

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla\psi) = \mathbf{Q}^\psi \tag{5}$$

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PDE: Definition

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = Q^{\psi} \qquad (6)$$

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, ..., \frac{\partial^n \psi}{\partial x_i^n}) = 0 \qquad , \qquad i = 3 \qquad (7)$$

where L is s differential operator.

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PDE: Klassifikation

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = Q^{\psi} \quad (8)$$

Physical problem	Math. problem	Examples
Equilibrium problems	Elliptic equations	Irrotational incompressible flow Inviscid incompressible flow Steady state heat conduction
Propagation problems (infinite propagation speed)	Parabolic equations	Unsteady viscous flow Transient heat transfer
Propagation problems (finite propagation speed)	Hyperbolic equations	Wave propagation (vibration) Inviscid supersonic flow

- Parabolisch: Diffusion, Gerinne (nichtlinear)
- Elliptisch: Grundwasser (stationär)

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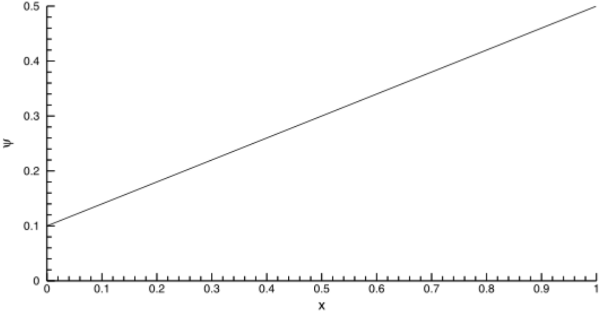
PDE: Elliptic Equation 1-D

$$\frac{d^2\psi}{dx^2} = 0 \tag{9}$$

$$\psi = ax + b \tag{10}$$



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PDE: Elliptic Equation 2-D

The prototype of an elliptic equation is the Laplace equation.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{11}$$

By substitution it can be easily verified that the exact solution of the Laplace equation is

$$\psi = sin(\pi x)exp(-\pi y) \tag{12}$$

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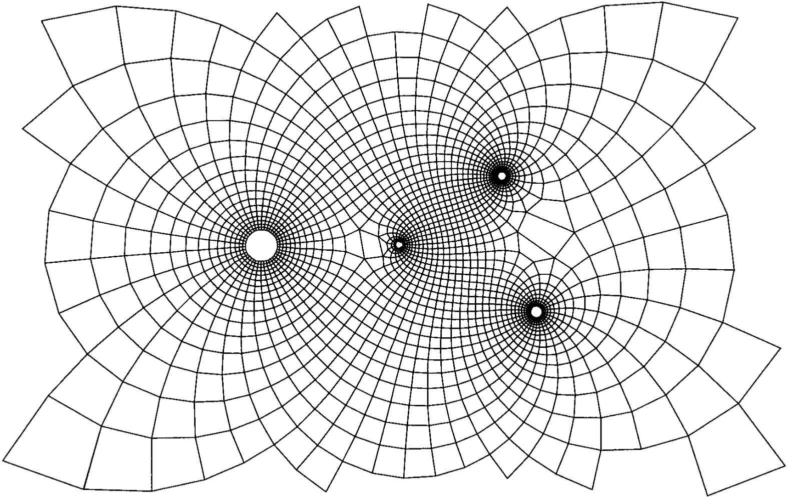
PDE: Elliptic Equation 2-D



PDE: Parabolic Equation 1-D

$$\frac{\partial \psi}{\partial t} = \alpha \frac{\partial^2 \psi}{\partial x^2} \tag{13}$$

Multiple solutions:

$$\psi(t, x) = sin(\sqrt{\pi\alpha}x)exp(-\pi t) \quad ??? \tag{14}$$

$$\psi(t, x) = sin(\frac{\pi}{\sqrt{\alpha}}x)exp(-\pi^2 t) \tag{15}$$

$$\psi(t, x) = sin(\pi x)exp(-\alpha\pi^2 t) \tag{16}$$

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PDE: Parabolic Equation 1-D

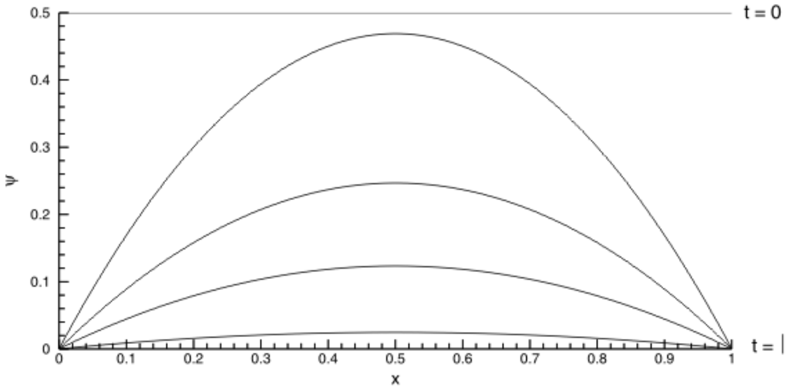

Figure: Solution of a parabolic equation

PDE: Hyperbolic Equation 1-D

$$\frac{\partial^2 \psi}{\partial^2 t} - c^2 \frac{\partial^2 \psi}{\partial^2 x} = 0 \tag{17}$$

$$\psi(t, x) = a \cos(\frac{\pi c t}{L}) \sin(\frac{\pi x}{L}) \tag{18}$$

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PDE: Hyperbolic Equation 1-D - Characteristics

$$A\frac{\partial \psi}{\partial t} - B\frac{\partial \psi}{\partial x} = 0 \tag{19}$$

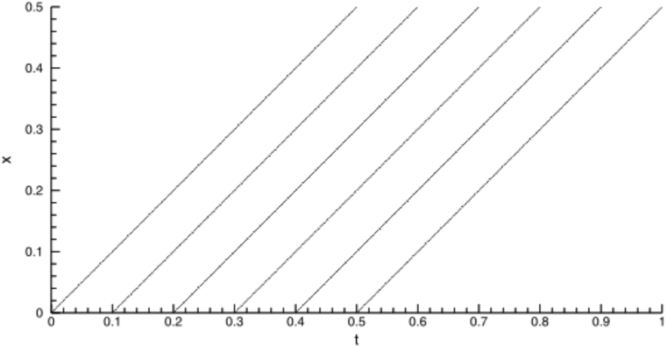

Figure: Characteristics of a hyperbolic equation

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PDE: Hyperbolic Equation 1-D - Characteristics

$$A\frac{\partial\psi}{\partial t} - B\frac{\partial\psi}{\partial x} = 0 \tag{20}$$

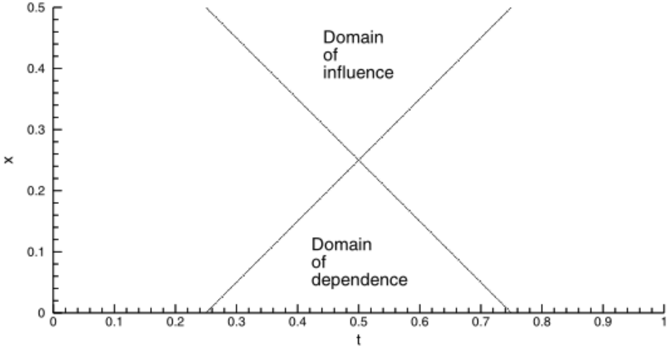

Figure: Domains of a hyperbolic equation

PDE: Equation Types

The following table gives typical examples of balance equations for the denoted quantities and their PDE types.

Physics	Equation structure	Examples
Continuity	$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$	Laplace equation
Mass/energy	$\frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} - \alpha \frac{\partial^2 \psi}{\partial x^2} = 0$	Fokker-Planck equation
Momentum	$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x}[\alpha(\psi) \frac{\partial \psi}{\partial x}] = 0$	Navier-Stokes equation

Boundary Conditions I

The following table gives an overview on common boundary condition types and its mathematical representation.

Table: Boundary conditions types

Type of BC	Mathematical Meaning	Physical Meaning
Dirichlet type	ψ	prescribed value potential surface
Neumann type	$\nabla\psi$	prescribed flux stream surface
Cauchy type	$\psi + A\nabla\psi$	resistance between potential and stream surface

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Boundary Conditions II

To describe conditions at boundaries we can use flux expressions of conservation quantities.

Table: Fluxes through surface boundaries

Quantity	Flux term
Mass	$\rho \mathbf{v}$
Momentum	$\rho \mathbf{v}\mathbf{v} - \sigma$
Energy	$\rho e \mathbf{v} - \lambda \nabla \mathbf{T}$

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Aufgabe: Prüfen sie die Gültigkeit der Lösungen für die partiellen Differentialgleichungen: (12), (14), (15), (16), (18).

Lösungsweg: Berechnen sie hierfür die entsprechenden partiellen Ableitungen und setzen sie diese dann in die entsprechenden Gleichungen (12), (14), (15), (16), (18) ein.

$$\frac{\partial \psi}{\partial t} = \ldots \quad , \quad \frac{\partial \psi}{\partial x} = \ldots \tag{21}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \ldots \quad , \quad \frac{\partial^2 \psi}{\partial y^2} = \ldots \tag{22}$$

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