

Hydroinformatik II

”Prozesssimulation und Systemanalyse”

HyBHW-1-02-03 @ 2021

Grundlagen der Hydromechanik

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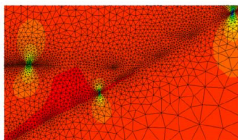
²Centre for Advanced Water Research – CAWR

30.04.2021 - Dresden

Zeitplan: Hydroinformatik II

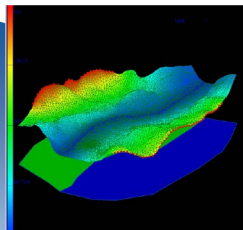
Datum	V	Thema	T
16.04.2021	01	Einführung in die Lehrveranstaltung Tools	L
23.04.2021	02	Grundlagen: Kontinuumsmechanik	L
30.04.2021	03	Grundlagen: Hydromechanik	L
07.05.2021	04	Grundlagen: Partielle Partialgleichungen	L
14.05.2021	05a	Tools: Compiler, Python, Jupyter	E
14.05.2021	05b	Übung: Elliptische PDG	E
21.05.2021	06	Übungen: Übersicht und Werkzeuge	E
28.05.2021	–	Pfingsten	
04.06.2021	07	Grundlagen: Näherungsverfahren	L
11.06.2021	08	Numerik: Finite-Differenzen-Methode (explizit)	L
18.06.2021	09	Numerik: Finite-Differenzen-Methode (implizit)	L
25.06.2021	10	Reserve	L/E
02.07.2021	11	Übung: Diffusionsprozesse	E
09.07.2021	12	Übung: Gerinnehydraulik	E
16.07.2021	13	Übung: Grundwassermodellierung	E
23.07.2021	14	Beleg: Besprechung zur Vorbereitung	L

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$

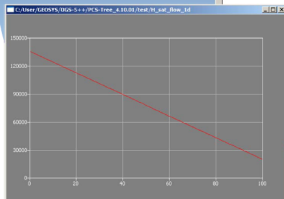
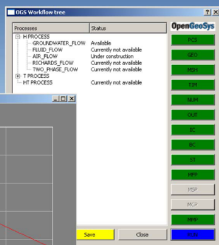


Basics
Mechanik

Anwendung



Numerische
Methoden



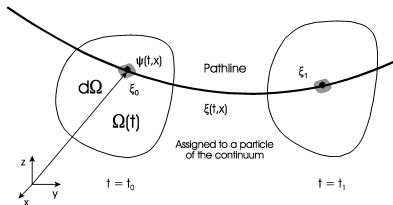
Programmierung
Visual C++

Prozessverständnis

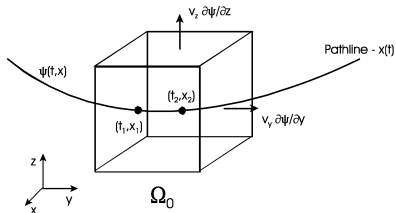
- ▶ Erhaltungsgrößen
- ▶ Massenerhaltung
- ▶ Fluidmassenerhaltung
- ▶ Diffusion
- ▶ Impulserhaltung
- ▶ Spannungen
- ▶ Fluiddruck
- ▶ Strömungsprobleme

General Balance Equation

Lagrange



Euler



General Balance Equation

$$\begin{aligned}\frac{d}{dt} \int_{\Omega} \psi d\Omega &= \frac{\partial}{\partial t} \int_{\Omega} \psi d\Omega + \oint_{\partial\Omega} \boldsymbol{\Phi}^{\psi} \cdot d\mathbf{S} \\ &= \frac{\partial}{\partial t} \int_{\Omega} \psi d\Omega + \int_{\Omega} \nabla \cdot \boldsymbol{\Phi}^{\psi} d\Omega\end{aligned}\quad (1)$$

$\lim d\Omega \rightarrow 0$

$$\begin{aligned}\frac{d\psi}{dt} &= \frac{\partial\psi}{\partial t} + \nabla \cdot \boldsymbol{\Phi}^{\psi} \\ &= \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla\psi) \\ &= Q^{\psi}\end{aligned}\quad (2)$$

HA 02#2020: Komponentenschreibweise $\nabla \cdot (\mathbf{v}\psi)$, $\nabla \cdot (\mathbf{D}^{\psi} \nabla\psi)$

Conservation Quantities (1.1.2)

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \quad (3)$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ , momentum density $\rho\mathbf{v}$ or energy density e .

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
Linear momentum	\mathbf{m}	Linear momentum density	$\rho\mathbf{v}$
Energy	E	Energy density	$e = \rho i + \frac{1}{2}\rho v^2$

(Phase) Mass Conservation

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla\psi) = Q^\psi \quad (4)$$

The differential equation of mass conservation in divergence form becomes

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \quad (5)$$

Partial differentiation of the above equation gives

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \mathbf{v} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{v} = 0 \quad (6)$$

(Phase) Mass Conservation

Using the material (or convective) derivative the mass conservation equation can be rewritten as

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{v} \quad (7)$$

Note, above convective form of mass conservation equation becomes zero only for incompressible flows, i.e.

$$\frac{\partial \rho}{\partial t} = 0 \quad (8)$$

requires divergence-free flow.

$$\nabla \cdot \mathbf{v} = 0 \quad (9)$$

From eqn. (6) results that the above expression is the continuity equation for a homogeneous fluid ($\rho = \text{const}$).

$$\nabla \cdot \mathbf{v} = 0 \quad (10)$$

Links:

- ▶ <https://de.wikipedia.org/wiki/Stromfunktion>
- ▶ <https://www.ingenieurkurse.de/stroemungslehre/ebene-stroemungen/quelle-und-senke-divergenz.html>

Conservation Quantities (1.1.2)

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \quad (11)$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ , momentum density $\rho\mathbf{v}$ or energy density e .

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
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Momentum Conservation

$$\psi = \rho \mathbf{v}$$

$$\frac{\partial}{\partial t} \int_{\Omega} \rho \mathbf{v} d\Omega + \oint_{\partial\Omega} \boldsymbol{\Phi}^m \cdot d\mathbf{S} = \int_{\Omega} \rho \mathbf{f} d\Omega \quad (12)$$

Flux term: The advective momentum flux is defined as

$$\boldsymbol{\Phi}^m = (\rho \mathbf{v}) \otimes \mathbf{v} = (\rho \mathbf{v}) \mathbf{v} \quad (13)$$

$$\mathbf{F} = \int_{\Omega} \rho \mathbf{f} d\Omega = \int_{\Omega} \rho (\mathbf{f}^e + \mathbf{f}^i) d\Omega = \underbrace{\int_{\Omega} \rho \mathbf{f}^e d\Omega}_{\text{External forces}} + \underbrace{\oint_{\partial\Omega} \boldsymbol{\sigma} : d\mathbf{S}}_{\text{Internal forces}} \quad (14)$$

Momentum Conservation

Substituting now flux and source terms of momentum we obtain

$$\frac{\partial}{\partial t} \int_{\Omega} \rho \mathbf{v} d\Omega + \oint_{\partial\Omega} \rho \mathbf{v} (\mathbf{v} \cdot d\mathbf{S}) = \int_{\Omega} \rho \mathbf{f}^e d\Omega + \oint_{\partial\Omega} \boldsymbol{\sigma} d\mathbf{S} \quad (15)$$

Applying the Gauss-Ostrogradskian theorem to the surface integrals

$$\begin{aligned} \oint_{\partial\Omega} \rho \mathbf{v} (\mathbf{v} \cdot d\mathbf{S}) &= \int_{\Omega} \nabla \cdot (\rho \mathbf{v} \mathbf{v}) d\Omega \\ \oint_{\partial\Omega} \boldsymbol{\sigma} : d\mathbf{S} &= \int_{\Omega} \nabla \cdot \boldsymbol{\sigma} d\Omega \end{aligned} \quad (16)$$

Momentum Conservation

The differential form of the momentum conservation law is then

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = \rho \mathbf{f}^e + \nabla \cdot \boldsymbol{\sigma} \quad (17)$$

The above equation is now extended by partial integration

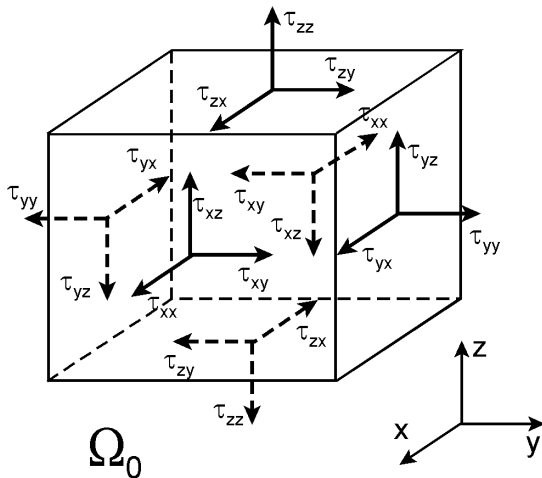
$$\begin{aligned} & \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t} + (\rho \mathbf{v}) \cdot \nabla \mathbf{v} + \mathbf{v} \nabla \cdot (\rho \mathbf{v}) \\ &= \rho \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] + \mathbf{v} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] \\ & \qquad \qquad \qquad = \rho \mathbf{f}^e + \nabla \cdot \boldsymbol{\sigma} \end{aligned} \quad (18)$$

Using the mass conservation equation (5) and dividing by ρ we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}^e + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} \quad (19)$$

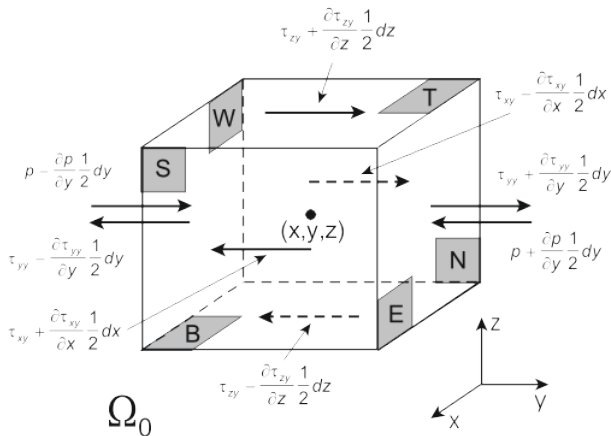
Momentum Conservation: Stress Tensor

$$\sigma = -p\mathbf{I} + \tau \quad , \quad \tau = \nu \nabla \mathbf{v} \quad (20)$$



Momentum Conservation: Stress Tensor

$$\boldsymbol{\tau} = \nu \nabla \mathbf{v} \quad (21)$$



$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}^e + \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} \quad (22)$$

In index notation the above vector equation is written as

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= \frac{1}{\rho} \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right) \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= \frac{1}{\rho} \left(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \right) \\ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= g + \frac{1}{\rho} \left(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \end{aligned} \quad (23)$$

with $u = v_x, v = v_y, w = v_z$ and $\mathbf{f}^e = \mathbf{g}$.

Flow Equations - Systematic

Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \quad (24)$$

Navier-Stokes Equation

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}^e - \frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v}} \quad (25)$$

Euler Equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}^e - \frac{1}{\rho} \nabla p \quad (26)$$

Stokes Equation

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{f}^e - \frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} \quad (27)$$

Darcy Equations

$$0 = \mathbf{f}^e - \frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v}$$

$$\psi = \rho_k = C_k \quad (29)$$

$$\frac{dC_k}{dt} = \frac{\partial C_k}{\partial t} + \nabla \cdot (\mathbf{v}C_k) - \nabla \cdot (\mathbf{D}_k \nabla C_k) = Q_k \quad (30)$$