Hydroinformatik II

"Prozesssimulation und Systemanalyse"

BHYWI-08-04 @ 2020

Partielle Differentialgleichungen

Olaf Kolditz

*Helmholtz Centre for Environmental Research – UFZ

[1] Technische Universität Dresden – TUDD

[2] Centre for Advanced Water Research – CAWR

15.05.2020 - Dresden

TECHNISCHE UNIVERSITÄT DRESDEN

Zeitplan: Hydroinformatik II

Datum	V	Thema	.
17.04.2020	00	Einführung in GoToMeeting (Web-Conferencing)	
17.04.2020	01	Einführung in die Lehrveranstaltung	
24.04.2020	02	Grundlagen: Kontinuumsmechanik	
08.05.2020	03	Grundlagen: Hydromechanik	
15.05.2020	04	Grundlagen: Partielle Partialgleichungen	
22.05.2020	05	Installation: Python, Qt C++	
29.05.2020	06	Programmierung: Einführung in Python	
05.06.2020	07	Numerik: Finite-Differenzen-Methode (explizit)	
12.06.2020	08	Numerik: Finite-Differenzen-Methode (implizit)	
19.06.2020	09	Anwendung: Diffusion (Übung)	
26.06.2020	10	Anwendung: Gerinnehydraulik (Theorie)	
03.07.2020	11	Anwendung: Gerinnehydraulik (Übung)	
10.07.2020	12	Anwendung: Grundwassermodellierung (datenbasierte Methoden)	
17.07.2020	13	Beleg: Besprechung zur Vorbereitung	

TECHNISCHE UNIVERSITÄT DRESDEN

Hausaufgabe 03/BHYWI04-2020 vom 15.05.2020

Partielle Differentialgleichungen

Aufgabe: Prüfen sie die Gültigkeit der Lösungen für die partiellen Differentialgleichungen: (17), (19), (20), (21), (23).

Lösungsweg: Berechnen sie hierfür die entsprechenden partiellen Ableitungen und setzen sie diese dann in die entsprechenden Gleichungen (17), (19), (20), (21), (23) ein.

$$\frac{\partial \psi}{\partial t} = \ldots \quad , \quad \frac{\partial \psi}{\partial x} = \ldots \tag{1}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \ldots \quad , \quad \frac{\partial^2 \psi}{\partial y^2} = \ldots \tag{2}$$

TECHNISCHE UNIVERSITÄT DRESDEN

Hausaufgabe 02/BHYWI02-2020 vom 08.05.2020

Tensoralegbra

$$\nabla \cdot (\mathbf{D}^\psi \nabla \psi) = \frac{\partial^2 (D_x^\psi \psi)}{\partial x^2} + \frac{\partial^2 (D_y^\psi \psi)}{\partial y^2} + \frac{\partial^2 (D_z^\psi \psi)}{\partial z^2} \qquad (3)$$

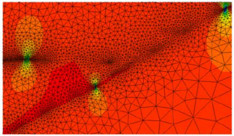
$\mathbf{D}^\psi$ ist ein (orthotroper) Tensor.

$$\mathbf{D}^\psi = \begin{pmatrix} D_x & 0 & 0 \\ 0 & D_y & 0 \\ 0 & 0 & D_z \end{pmatrix} \qquad (4)$$

TECHNISCHE UNIVERSITÄT DRESDEN

Konzept

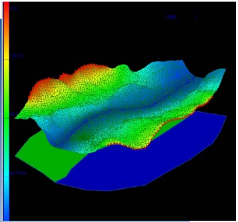
$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Basics

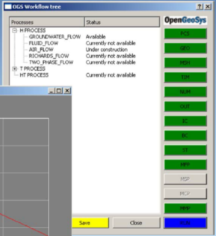
Mechanik

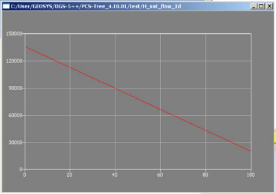
Anwendung



Numerische

Methoden





Programmierung

Visual C++

Prozessverständnis

Inhalte

- Konzept
- Partielle Differentialgleichungen
- Klassifikation
- Beispiele
- Anfangs- und Randbedingungen

Dimensionsanalyse

Navier-Stokes-Gleichung (jetzt käme eigentlich die Tafel dran ...)

Mathematical Classification (1.5)

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, ..., \frac{\partial^n \psi}{\partial x_i^n}) = 0 \qquad , \qquad i = 3 \tag{5}$$

where L is s differential operator. Second-order PDE with two independent variables are given by

$$A\frac{\partial^2 \psi}{\partial x^2} + B\frac{\partial^2 \psi}{\partial x \partial y} + C\frac{\partial^2 \psi}{\partial y^2} + D\frac{\partial \psi}{\partial x} + E\frac{\partial \psi}{\partial y} + F\psi + G = 0 \tag{6}$$

Second-order PDEs with more independent variables can be classified by examination of the eigenvalues of the matrix a_{ij}.

$$\sum_i \sum_j a_{ij} \frac{\partial \psi^2}{\partial x_i \partial x_j} + G = 0 \tag{7}$$

TECHNISCHE UNIVERSITÄT DRESDEN

Mathematical Classification (1.5)

PDE type	Discriminant	Eigenvalues	Canonical form	Example
Elliptic	$B^2 - 4AC < 0$ complex characteristics	$\forall\lambda > 0$ equal signs	$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0$	Laplace equation
Parabolic	$B^2 - 4AC = 0$	$\exists\lambda = 0$	$\frac{\partial^2\psi}{\partial\eta^2} = G$	Diffusion, Burgers equations
Hyperbolic	$B^2 - 4AC > 0$ real characteristics	$\exists\lambda < 0$ different signs	$\frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta^2} = 0$	Wave equation

TECHNISCHE UNIVERSITÄT DRESDEN

- Integral form

$$\int_\Omega \frac{d\psi}{dt} d\Omega = \int_\Omega \frac{\partial \psi}{\partial t} d\Omega + \int_\Omega \nabla \cdot (\mathbf{v}\psi) d\Omega - \int_\Omega \nabla \cdot (\mathbf{D}^\psi \nabla \psi) d\Omega = \int_\Omega Q^\psi d\Omega \tag{8}$$

- Differential form

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla \psi) = \mathbf{Q}^\psi \tag{9}$$

PDE: Definition

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla\cdot(\mathbf{v}\psi) - \nabla\cdot(\mathbf{D}^{\psi}\nabla\psi) = Q^{\psi} \tag{10}$$

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial\psi}{\partial x_i}, ..., \frac{\partial^n\psi}{\partial x_i^n}) = 0 \qquad , \qquad i = 3 \tag{11}$$

where L is s differential operator.

PDE: Klassifikation

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi}\nabla\psi) = Q^{\psi} \qquad (12)$$

Physical problem	Math. problem	Examples
Equilibrium problems	Elliptic equations	Irrotational incompressible flow Inviscid incompressible flow Steady state heat conduction
Propagation problems (infinite propagation speed)	Parabolic equations	Unsteady viscous flow Transient heat transfer
Propagation problems (finite propagation speed)	Hyperbolic equations	Wave propagation (vibration) Inviscid supersonic flow

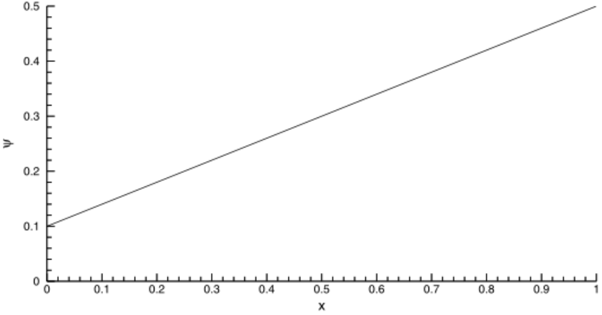
- Parabolisch: Diffusion, Gerinne (nichtlinear)
- Elliptisch: Grundwasser (stationär)

TECHNISCHE UNIVERSITÄT DRESDEN

PDE: Elliptic Equation 1-D

$$\frac{d^2\psi}{dx^2} = 0 \tag{13}$$

$$\psi = ax + b \tag{14}$$



PDE: Elliptic Equation 2-D

The prototype of an elliptic equation is the Laplace equation.

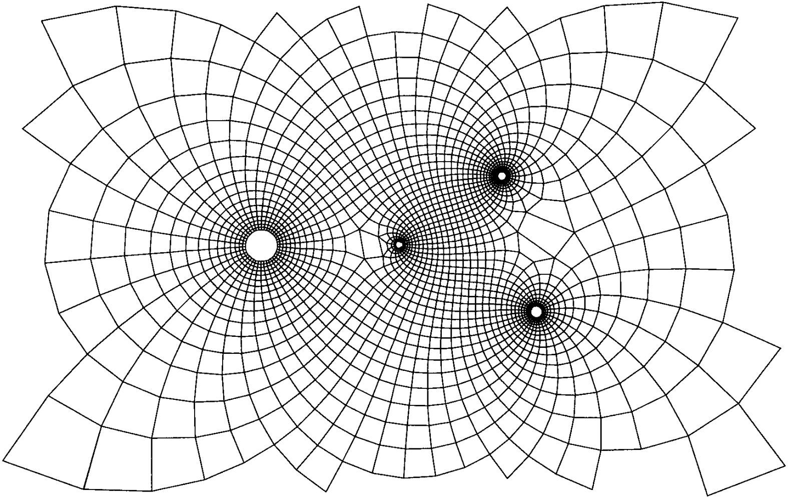
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{15}$$

By substitution it can be easily verified that the exact solution of the Laplace equation is

$$\psi = sin(\pi x)exp(-\pi y) \tag{16}$$

TECHNISCHE UNIVERSITÄT DRESDEN

PDE: Elliptic Equation 2-D



TECHNISCHE UNIVERSITÄT DRESDEN

PDE: Parabolic Equation 1-D

$$\frac{\partial \psi}{\partial t} = \alpha \frac{\partial^2 \psi}{\partial x^2} \tag{17}$$

Multiple solutions:

$$\psi(t, x) = sin(\sqrt{\pi\alpha}x)exp(-\pi t) \quad ??? \tag{18}$$

$$\psi(t, x) = sin(\frac{\pi}{\sqrt{\alpha}}x)exp(-\pi^2 t) \tag{19}$$

$$\psi(t, x) = sin(\pi x)exp(-\alpha\pi^2 t) \tag{20}$$

TECHNISCHE UNIVERSITÄT DRESDEN

PDE: Parabolic Equation 1-D

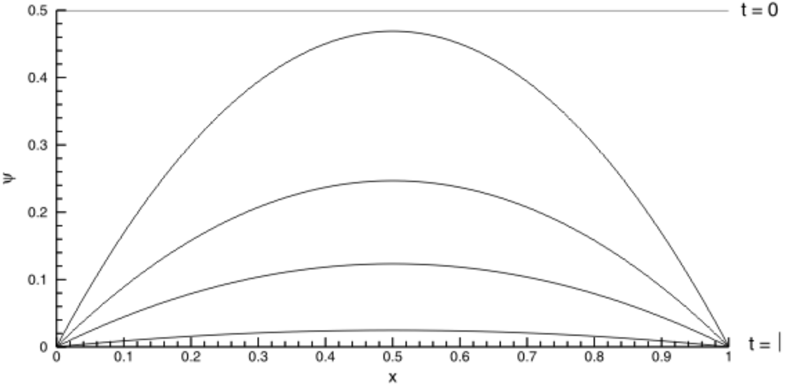


Fig.: Solution of a parabolic equation

TECHNISCHE UNIVERSITÄT DRESDEN

PDE: Hyperbolic Equation 1-D

$$\frac{\partial^2 \psi}{\partial^2 t} - c^2 \frac{\partial^2 \psi}{\partial^2 x} = 0 \tag{21}$$

$$\psi(t, x) = a \cos(\frac{\pi c t}{L}) \sin(\frac{\pi x}{L}) \tag{22}$$

TECHNISCHE UNIVERSITÄT DRESDEN

PDE: Hyperbolic Equation 1-D - Characteristics

$$A\frac{\partial \psi}{\partial t} - B\frac{\partial \psi}{\partial x} = 0 \qquad (23)$$

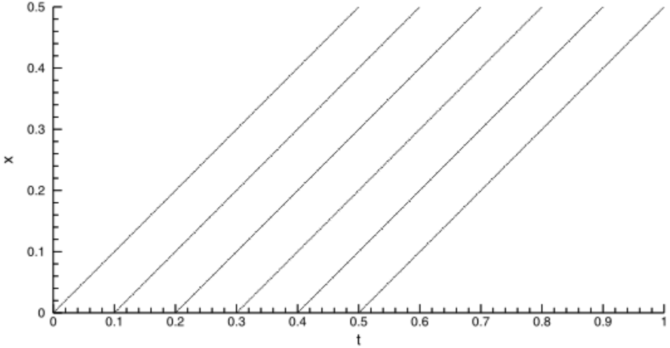


Fig.: Characteristics of a hyperbolic equation

$$A\frac{\partial\psi}{\partial t} - B\frac{\partial\psi}{\partial x} = 0 \tag{24}$$

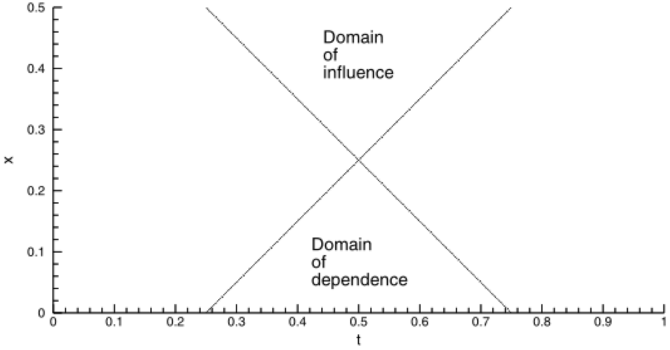


Fig.: Domains of a hyperbolic equation

PDE: Equation Types

The following table gives typical examples of balance equations for the denoted quantities and their PDE types.

Physics	Equation structure	Examples
Continuity	$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$	Laplace equation
Mass/energy	$\frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} - \alpha \frac{\partial^2 \psi}{\partial x^2} = 0$	Fokker-Planck equation
Momentum	$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x}[\alpha(\psi) \frac{\partial \psi}{\partial x}] = 0$	Navier-Stokes equation

Boundary Conditions I

The following table gives an overview on common boundary condition types and its mathematical representation.

Table: Boundary conditions types

Type of BC	Mathematical Meaning	Physical Meaning
Dirichlet type	ψ	prescribed value potential surface
Neumann type	$\nabla\psi$	prescribed flux stream surface
Cauchy type	$\psi + A\nabla\psi$	resistance between potential and stream surface

TECHNISCHE UNIVERSITÄT DRESDEN

Boundary Conditions II

To describe conditions at boundaries we can use flux expressions of conservation quantities.

Table: Fluxes through surface boundaries

Quantity	Flux term
Mass	$\rho \mathbf{v}$
Momentum	$\rho \mathbf{v}\mathbf{v} - \sigma$
Energy	$\rho e \mathbf{v} - \lambda \nabla \mathbf{T}$

TECHNISCHE UNIVERSITÄT DRESDEN