Hydroinformatik II

"Prozesssimulation und Systemanalyse"

BHYWI-08-03 @ 2020

Grundlagen der Hydromechanik

Olaf Kolditz

*Helmholtz Centre for Environmental Research – UFZ

[1] Technische Universität Dresden – TUDD

[2] Centre for Advanced Water Research – CAWR

08.05.2020 - Dresden

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Zeitplan: Hydroinformatik II

Datum	V	Thema	.
17.04.2020	00	Einführung in GoToMeeting (Web-Conferencing)	
17.04.2020	01	Einführung in die Lehrveranstaltung	
24.04.2020	02	Grundlagen: Kontinuumsmechanik	
08.05.2020	03	Grundlagen: Hydromechanik	
15.05.2020	04	Grundlagen: Partielle Partialgleichungen	
22.05.2020	05	Installation: Python, Qt C++	
29.05.2020	06	Programmierung: Einführung in Python	
05.06.2020	07	Numerik: Finite-Differenzen-Methode (explizit)	
12.06.2020	08	Numerik: Finite-Differenzen-Methode (implizit)	
19.06.2020	09	Anwendung: Diffusion (Übung)	
26.06.2020	10	Anwendung: Gerinnehydraulik (Theorie)	
03.07.2020	11	Anwendung: Gerinnehydraulik (Übung)	
10.07.2020	12	Anwendung: Grundwassermodellierung (datenbasierte Methoden)	
17.07.2020	13	Beleg: Besprechung zur Vorbereitung	

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Hausaufgabe vom 24.04.2020

Aufgabe: Was ist $\mathbf{v} \cdot \nabla\psi$?

$$\begin{aligned} \mathbf{v} &= (v_x, v_y, v_z) && (1) \\ \nabla\psi &= (\partial_x\psi, \partial_y\psi, \partial_z\psi) && (2) \\ \cdot &= \text{Skalarprodukt} && (3) \end{aligned}$$

$$\begin{aligned} (v_x, v_y, v_z) \cdot \begin{pmatrix} \partial_x\psi \\ \partial_y\psi \\ \partial_z\psi \end{pmatrix} &= v_x\partial_x\psi + v_y\partial_y\psi + v_z\partial_z\psi && (4) \\ &= v_x\frac{\partial\psi}{\partial_x} + v_y\frac{\partial\psi}{\partial_y} + v_z\frac{\partial\psi}{\partial_z} && (5) \end{aligned}$$

Ist die Gleichung (5) eine vektorielle oder skalare Größe ?

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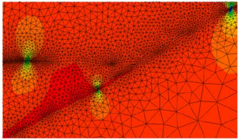
Konzept

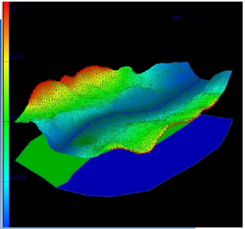
$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Basics

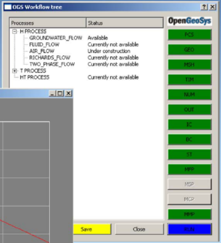
Mechanik

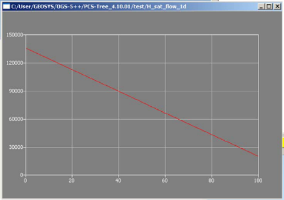
Anwendung



Numerische

Methoden





Programmierung

Visual C++

Prozessverständnis

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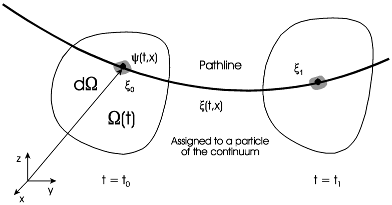
Inhalte

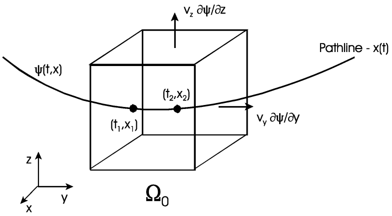
- Erhaltungsgrößen
- Massenerhaltung
- Fluidmassenerhaltung
- Diffusion
- Impulserhaltung
- Spannungen
- Fluiddruck
- Strömungsprobleme

General Balance Equation

Lagrange



Euler



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General Balance Equation

$$\frac{d}{dt}\int_\Omega \psi \, d\Omega = \frac{\partial}{\partial t}\int_\Omega \psi \, d\Omega + \oint_{\partial\Omega} \mathbf{\Phi}^\psi \cdot d\mathbf{S} \qquad (6)$$
$$= \frac{\partial}{\partial t}\int_\Omega \psi \, d\Omega + \int_\Omega \nabla \cdot \mathbf{\Phi}^\psi d\Omega$$

$\lim d\Omega \to 0$

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{\Phi}^\psi \qquad (7)$$
$$= \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla \psi)$$
$$= Q^\psi$$

HA 02#2020: Komponentenschreibweise $\nabla \cdot (\mathbf{v}\psi)$, $\nabla \cdot (\mathbf{D}^\psi \nabla \psi)$

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Conservation Quantities (1.1.2)

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \tag{8}$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ, momentum density $\rho\mathbf{v}$ or energy density e.

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
Linear momentum	$\mathbf{m}$	Linear momentum density	$\rho\mathbf{v}$
Energy	E	Energy density	$e = \rho i + \frac{1}{2}\rho v^2$

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(Phase) Mass Conservation

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = Q^{\psi} \tag{9}$$

The differential equation of mass conservation in divergence form becomes

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \tag{10}$$

Partial differentiation of the above equation gives

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0 \tag{11}$$

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(Phase) Mass Conservation

Using the material (or convective) derivative the mass conservation equation can be rewritten as

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{v} \tag{12}$$

Note, above convective form of mass conservation equation becomes zero only for incompressible flows, i.e.

$$\frac{\partial \rho}{\partial t} = 0 \tag{13}$$

requires divergence-free flow.

$$\nabla \cdot \mathbf{v} = 0 \tag{14}$$

From eqn. (11) results that the above expression is the continuity equation for a homogeneous fluid ($\rho = \text{const}$).

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Divergenzfreie Strömung

$$\nabla \cdot \mathbf{v} = 0 \tag{15}$$

Links:

- https://de.wikipedia.org/wiki/Stromfunktion
- https://www.ingenieurkurse.de/stroemungslehre/ebene-stroemungen/quelle-und-senke-divergenz.html

Conservation Quantities (1.1.2)

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \tag{16}$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ, momentum density $\rho\mathbf{v}$ or energy density e.

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
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Momentum Conservation

$\psi = \rho \mathbf{v}$

$$\frac{\partial}{\partial t} \int_{\Omega} \rho \mathbf{v} d\Omega + \oint_{\partial\Omega} \mathbf{\Phi}^m \cdot d\mathbf{S} = \int_{\Omega} \rho \mathbf{f} d\Omega \qquad (17)$$

Flux term: The advective momentum flux is defined as

$$\mathbf{\Phi}^m = (\rho \mathbf{v}) \otimes \mathbf{v} = (\rho \mathbf{v}) \mathbf{v} \qquad (18)$$

$$\mathbf{F} = \int_{\Omega} \rho \mathbf{f} d\Omega = \int_{\Omega} \rho (\mathbf{f}^e + \mathbf{f}^i) d\Omega = \underbrace{\int_{\Omega} \rho \mathbf{f}^e d\Omega}_{\text{External forces}} + \underbrace{\oint_{\partial\Omega} \sigma : \mathbf{dS}}_{\text{Internal forces}} \qquad (19)$$

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Momentum Conservation

Substituting now flux and source terms of momentum we obtain

$$\frac{\partial}{\partial t}\int\limits_{\Omega} \rho \mathbf{v} d\Omega + \oint\limits_{\partial\Omega} \rho \mathbf{v}(\mathbf{v}\cdot d\mathbf{S}) = \int\limits_{\Omega} \rho \mathbf{f}^{e} d\Omega + \oint\limits_{\partial\Omega} \boldsymbol{\sigma} : d\mathbf{S} \tag{20}$$

Applying the Gauss-Ostrogradskian theorem to the surface integrals

$$\begin{aligned} \oint\limits_{\partial\Omega} \rho \mathbf{v}(\mathbf{v}\cdot d\mathbf{S}) &= \int\limits_{\Omega} \nabla\cdot(\rho\mathbf{v}\mathbf{v})\, d\Omega \\ \oint\limits_{\partial\Omega} \boldsymbol{\sigma} : d\mathbf{S} &= \int\limits_{\Omega} \nabla\cdot\boldsymbol{\sigma}\, d\Omega \end{aligned} \tag{21}$$

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Momentum Conservation

The differential form of the momentum conservation law is then

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \rho \mathbf{f}^e + \nabla \cdot \boldsymbol{\sigma} \tag{22}$$

The above equation is now extended by partial integration

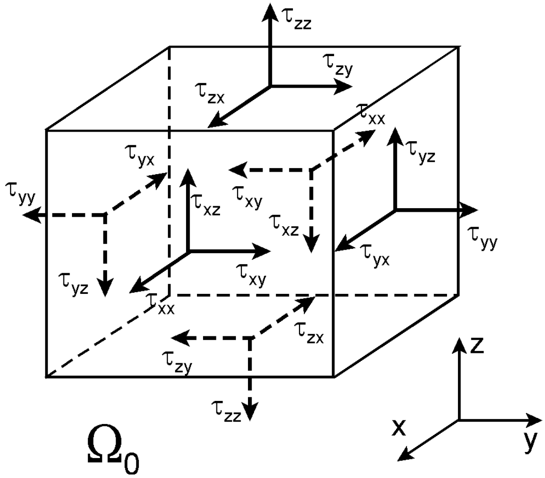
$$\begin{aligned} \rho\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t} + (\rho \mathbf{v}) \cdot \nabla \mathbf{v} + \mathbf{v}\nabla \cdot (\rho \mathbf{v}) \\ = \rho[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}] + \mathbf{v}[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})] \\ = \rho \mathbf{f}^e + \nabla \cdot \boldsymbol{\sigma} \end{aligned} \tag{23}$$

Using the mass conservation equation (10) and dividing by ρ we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} \tag{24}$$

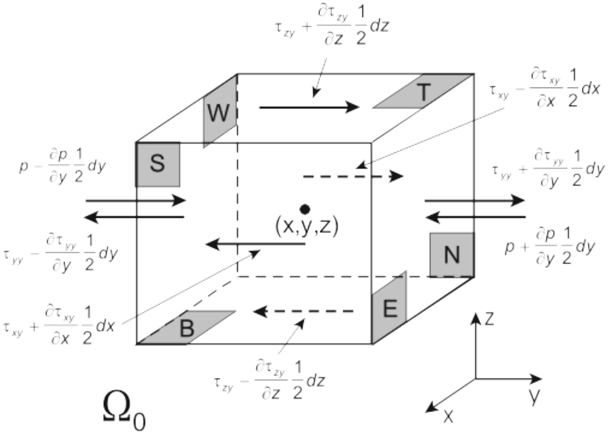
Momentum Conservation: Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \quad , \quad \boldsymbol{\tau} = \nu \nabla \mathbf{v} \tag{25}$$



Momentum Conservation: Stress Tensor

$$\boldsymbol{\tau} = \nu \nabla \mathbf{v} \tag{26}$$



Fluid Momentum Balance

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} \qquad (27)$$

In index notation the above vector equation is written as

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \frac{1}{\rho}(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}) \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= \frac{1}{\rho}(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}) \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= g + \frac{1}{\rho}(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z})
\end{aligned} \qquad (28)$$

with $u = v_x, v = v_y, w = v_z$ and $\mathbf{f}^e = \mathbf{g}$.

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Flow Equations - Systematic

Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \tag{29}$$

Navier-Stokes Equation

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v}} \tag{30}$$

Euler Equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p \tag{31}$$

Stokes Equation

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} \tag{32}$$

Darcy Equations

$$0 = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} \tag{33}$$

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(Component) Mass Conservation

$$\psi = \rho_k = C_k \tag{34}$$

$$\frac{dC_k}{dt} = \frac{\partial C_k}{\partial t} + \nabla \cdot (\mathbf{v} C_k) - \nabla \cdot (\mathbf{D_k} \nabla C_k) = Q_k \tag{35}$$

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