

# BHYWI-08: Semester-Fahrplan 2019

## Vorlesungen

Datum	V	Thema
05.04.2019	01	IT: GitHub / Qt Installation
12.04.2019	02	Grundlagen: Kontinuumsmechanik
19.04.2019	--	Ostern
26.04.2019	03	Grundlagen: Hydromechanik
03.05.2019	04	Grundlagen: Partielle Differentialgleichungen
10.05.2019	05	Grundlagen: Numerik, Qt Übung: Funktionsrechner
17.05.2019	06	Numerik: Finite Differenzen Methode I (explizit)
24.05.2019	07	Numerik: Finite Differenzen Methode II (implizit)
31.05.2019	08	Gerinnehydraulik: Theorie – Grundlagen
07.06.2019	09	Gerinnehydraulik: Programmierung, Übung
14.06.2019		Pfingsten
21.06.2019	10	Grundwassermodellierung: Catchment Übung
28.06.2019	11	Grundwassermodellierung: Datenbasierte Methoden I
05.07.2019	12	Grundwassermodellierung: Datenbasierte Methoden II
12.07.2019	13	Beleg

# Hydroinformatik II

## "Prozesssimulation und Systemanalyse"

### BHYWI-08-06 © 2019

### Finite-Differenzen-Methode

Olaf Kolditz

\*Helmholtz Centre for Environmental Research – UFZ

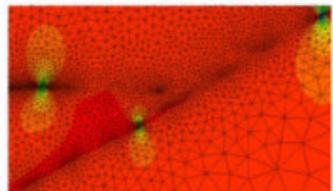
<sup>1</sup>Technische Universität Dresden – TUDD

<sup>2</sup>Centre for Advanced Water Research – CAWR

17.05.2019 - Dresden

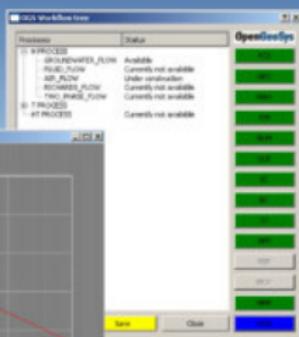
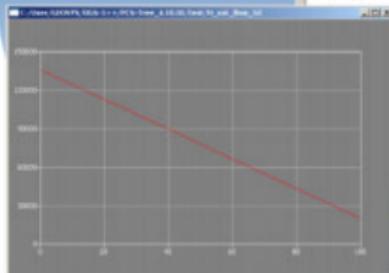
# Konzept

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \mathbf{v}^E \nabla \psi$$

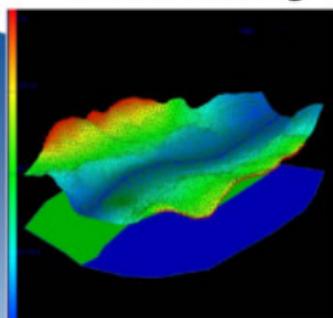


Basics  
Mechanik

Numerische  
Methoden



Anwendung



Programmierung  
Visual C++

Prozessverständnis

## Übung

- ▶ Python ...
- 

## Vorlesung

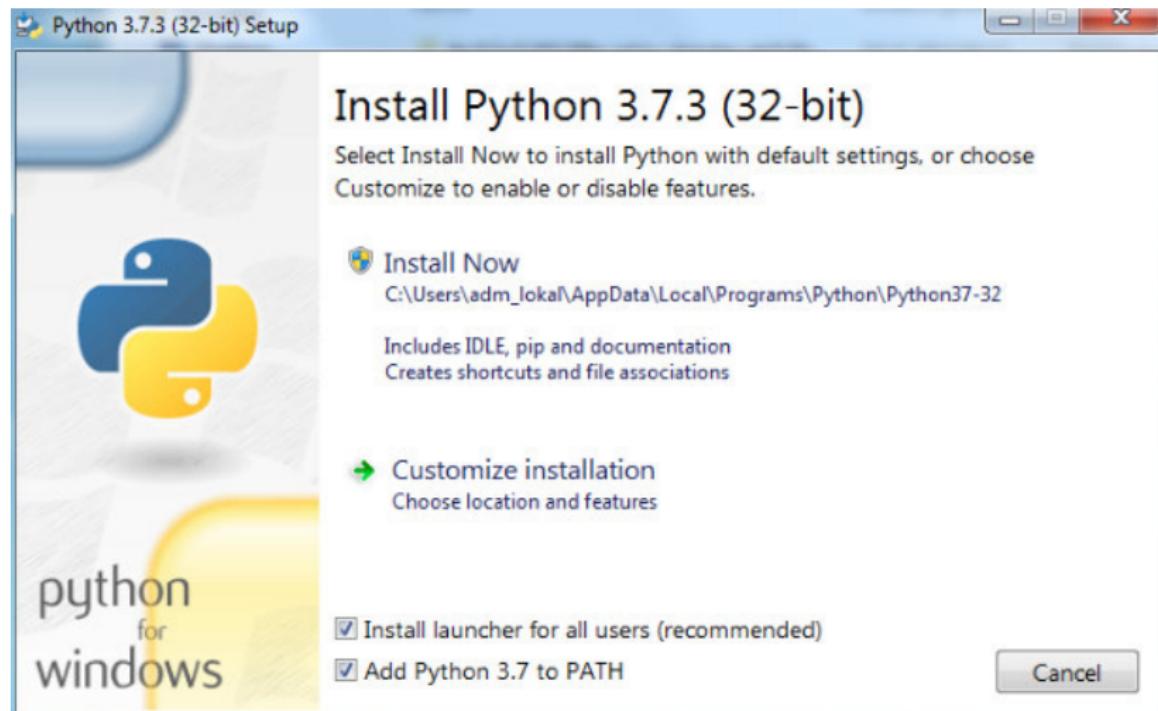
- ▶ Grundlagen der Finite Differenzen Methode
- ▶ Approximation methods
- ▶ Finite difference method – FDM (Ch. 3)
- ▶ Taylor series expansion
- ▶ Derivatives
- ▶ Diffusion equation
- ▶ (Finite element method – FEM ⇒ Hydrosystemanalyse)

# Python, die Zweite (Path)

<https://www.python.org/downloads/>

The screenshot shows a Microsoft Edge browser window displaying the Python Software Foundation's download page. The URL in the address bar is <https://www.python.org/downloads/>. The page features a large Python logo at the top left. A prominent yellow button labeled "Download Python 3.7.3" is centered above a section titled "Download the latest version for Windows". To the right of this section is a graphic of two parachutes descending from the sky, each carrying a package. Below the download button, there are links for other operating systems: "Looking for Python with a different OS? Python for [Windows](#), [Linux/UNIX](#), [Mac OS X](#), [Other](#)". Further down, there are links for "Pre-releases" and "Docker images". A yellow banner at the bottom encourages users to "Contribute to the PSF by Purchasing a PyCharm License. All proceeds benefit the PSF." A "Donate Now" button is also present in this banner. At the very bottom of the page, there is a table with columns for "Release version", "Release date", and "Click for more".

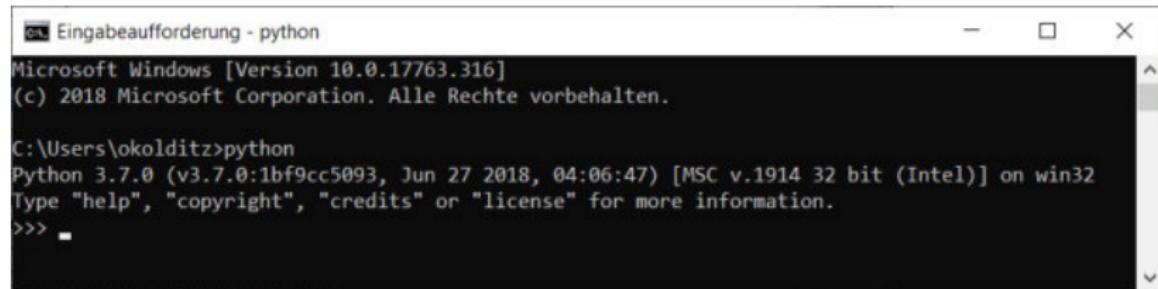
# Python, die Zweite (Path)



# Python, die Zweite (Path)

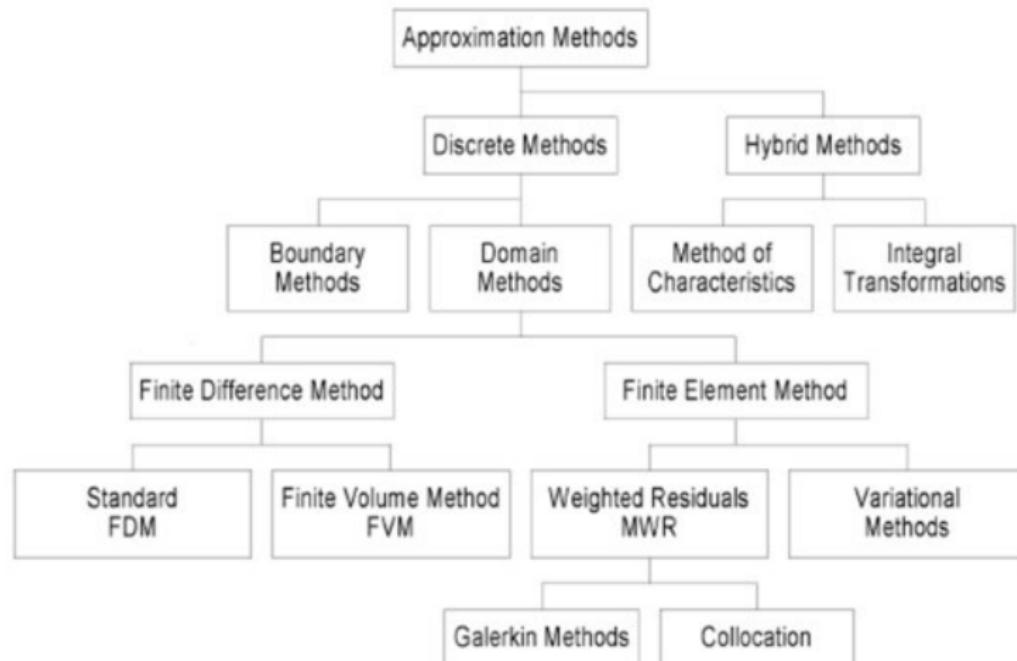
Zwei Optionen:

- 1 Python deinstallieren, neu installieren und "Add Python to PATH"
- 2 "PATH" nachträglich ergänzen (unterschiedlich für verschiedene Windows-Versionen), am besten googeln

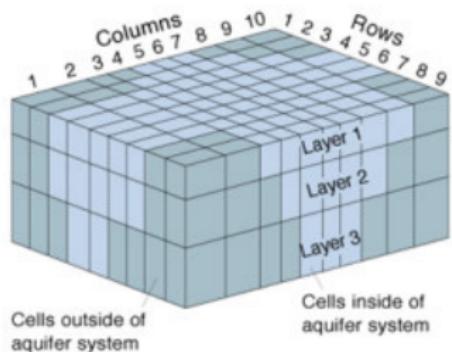


```
Eingabeaufforderung - python
Microsoft Windows [Version 10.0.17763.316]
(c) 2018 Microsoft Corporation. Alle Rechte vorbehalten.

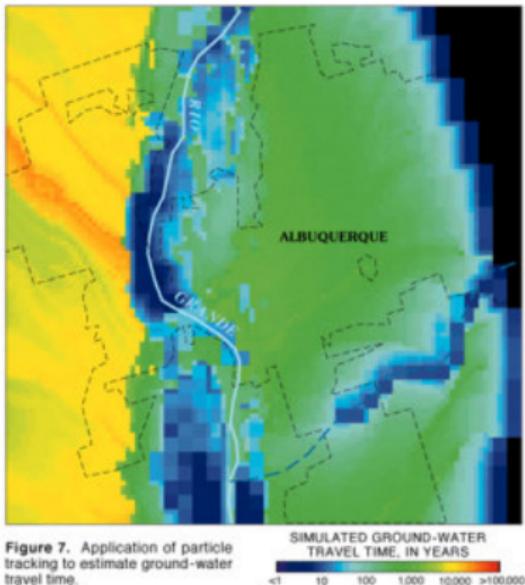
C:\Users\okolditz>python
Python 3.7.0 (v3.7.0:1bf9cc5093, Jun 27 2018, 04:06:47) [MSC v.1914 32 bit (Intel)] on win32
Type "help", "copyright", "credits" or "license" for more information.
>>> -
```



# FDM Anwendungen - MODFLOW



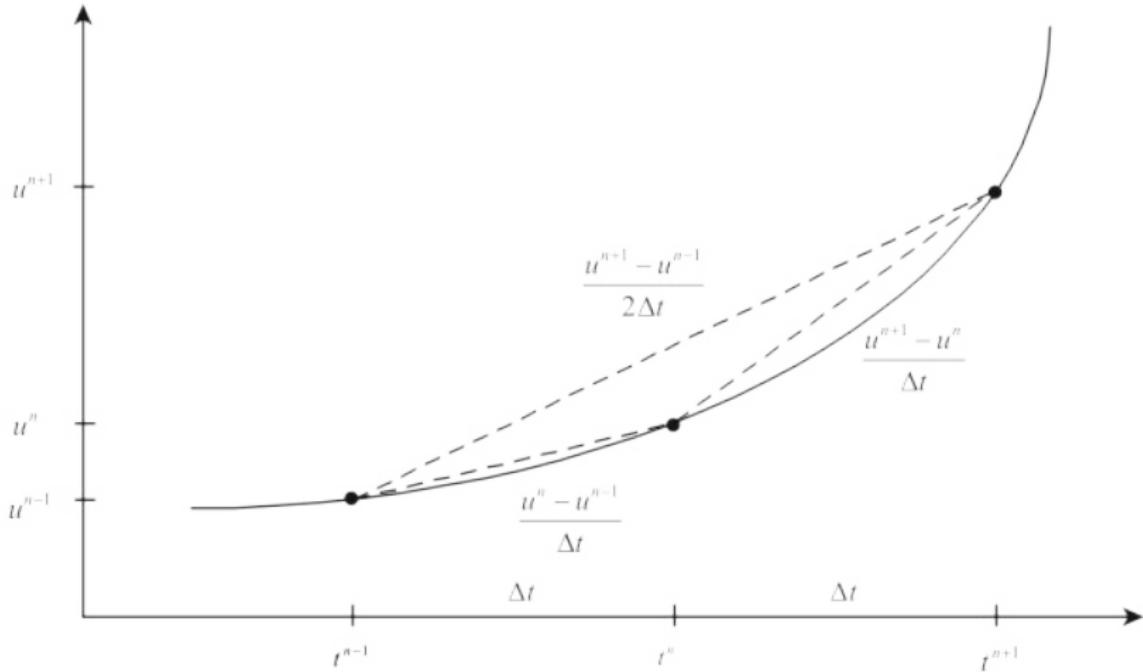
**Figure 2.** Example of model grid for simulating three-dimensional ground-water flow.

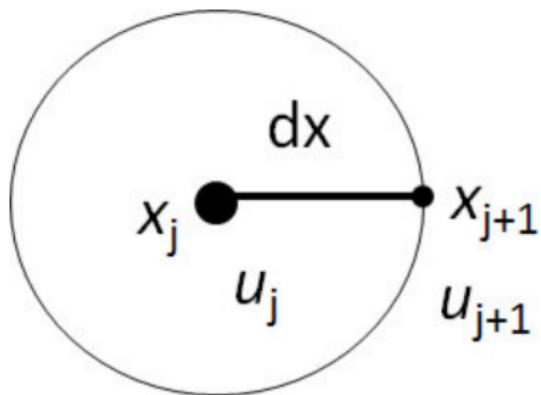


**Figure 7.** Application of particle tracking to estimate ground-water travel time.

<http://water.usgs.gov/pubs/FS/FS-121-97/images/fig7.gif>

# Ableitungen





in time

$$u_j^{n+1} = \sum_{m=0}^{\infty} \frac{\Delta t^m}{m!} \left[ \frac{\partial^m u}{\partial t^m} \right]_j^n \quad (1)$$

$$\Delta t = t^{n+1} - t^n$$

in space

$$u_{j+1}^n = \sum_{m=0}^{\infty} \frac{\Delta x^m}{m!} \left[ \frac{\partial^m u}{\partial x^m} \right]_j^n \quad (2)$$

$$\Delta x = x_{j+1} - x_j$$

$$u_j^{n+1} = u_j^n + \Delta t \left[ \frac{\partial u}{\partial t} \right]_j^n + \frac{\Delta t^2}{2} \left[ \frac{\partial^2 u}{\partial t^2} \right]_j^n + O(\Delta t^3) \quad (3)$$

$$u_{j+1}^n = u_j^n + \Delta x \left[ \frac{\partial u}{\partial x} \right]_j^n + \frac{\Delta x^2}{2} \left[ \frac{\partial^2 u}{\partial x^2} \right]_j^n + O(\Delta x^3) \quad (4)$$

# 1. Ableitung

$$\left[ \frac{\partial u}{\partial t} \right]_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\Delta t}{2} \left[ \frac{\partial^2 u}{\partial t^2} \right]_j^n + o(\Delta t^2) \quad (5)$$

$$\left[ \frac{\partial u}{\partial x} \right]_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} - \frac{\Delta x}{2} \left[ \frac{\partial^2 u}{\partial x^2} \right]_j^n + o(\Delta x^2) \quad (6)$$

Forward difference approximation

$$\left[ \frac{\partial u}{\partial x} \right]_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} + O(\Delta x) \quad (7)$$

Backward difference approximation

$$\left[ \frac{\partial u}{\partial x} \right]_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x} + O(\Delta x) \quad (8)$$

Central difference approximation

$$\left[ \frac{\partial u}{\partial x} \right]_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O(\Delta x^2) \quad (9)$$

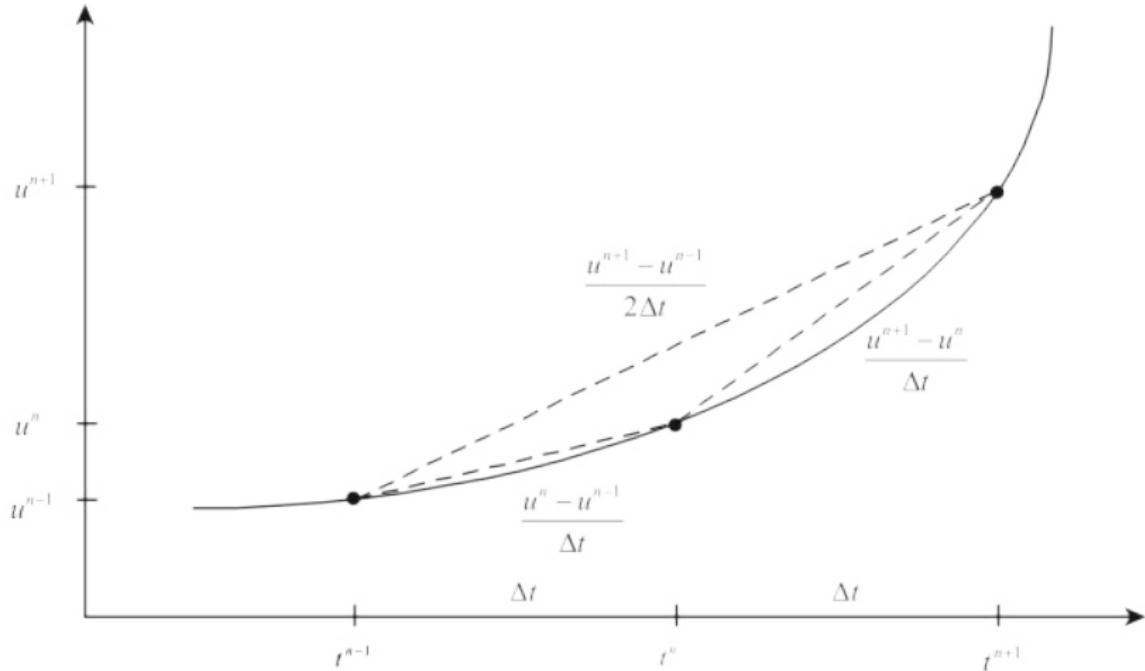
# Zentrale Differenzen

$$\begin{aligned} u_{j+1}^n &= u_j^n + \Delta x \left[ \frac{\partial u}{\partial x} \right]_j^n + \frac{\Delta x^2}{2} \left[ \frac{\partial^2 u}{\partial x^2} \right]_j^n + O(\Delta x^3) \\ u_{j-1}^n &= u_j^n - \Delta x \left[ \frac{\partial u}{\partial x} \right]_j^n + \frac{\Delta x^2}{2} \left[ \frac{\partial^2 u}{\partial x^2} \right]_j^n - O(\Delta x^3) \quad (10) \end{aligned}$$

Central difference approximation

$$\left[ \frac{\partial u}{\partial x} \right]_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O(\Delta x^2) \quad (11)$$

# Ableitungen

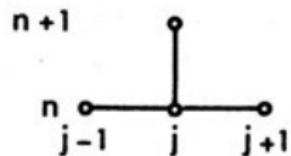


## 2. Ableitung

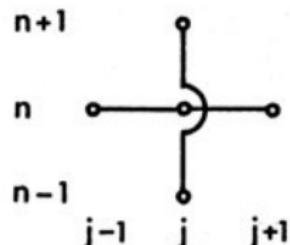
$$\begin{aligned}\left[ \frac{\partial^2 u}{\partial x^2} \right]_j^n &\approx \frac{1}{\Delta x} \left( \left[ \frac{\partial u}{\partial x} \right]_{j+1}^n - \left[ \frac{\partial u}{\partial x} \right]_j^n \right) \\ &\approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\end{aligned}\tag{12}$$

$$\left[ \frac{\partial^2 u}{\partial x^2} \right]_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{\Delta x^2}{12} \left[ \frac{\partial^4 u}{\partial x^4} \right]_j^n + \dots\tag{13}$$

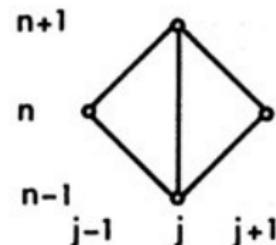
# Übersicht Differenzenverfahren



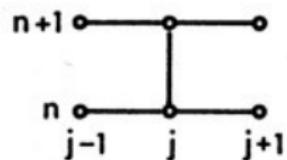
FTCS



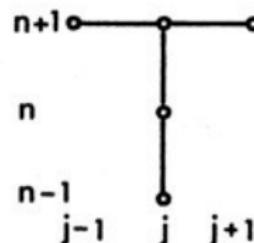
Richardson



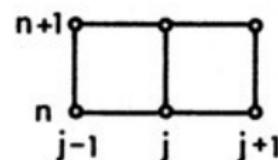
Du Fort-Frankel



Crank-Nicolson



3LFI



Linear F.E.M./  
Crank-Nicolson

# Diffusionsgleichung

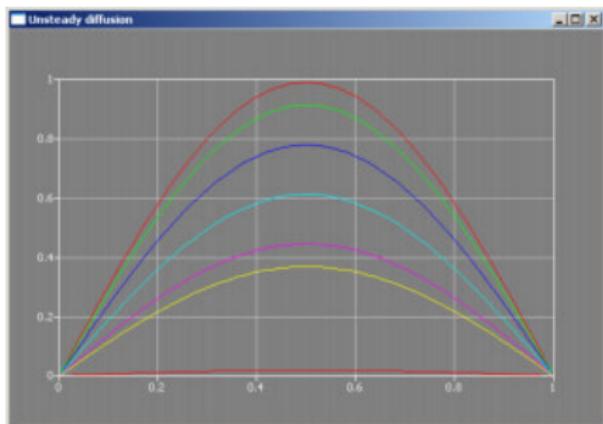
$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (14)$$

- ▶ Diffusion equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (15)$$

- ▶ Analytical solution

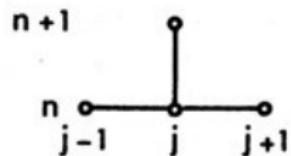
$$u = \sin(\pi x) e^{-\alpha t^2} \quad (16)$$



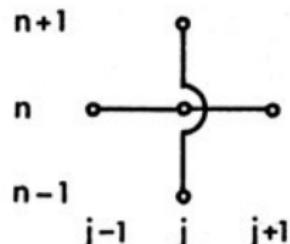
⇒ Übung

- ▶ K: validity

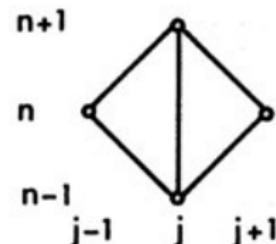
# Übersicht Differenzenverfahren



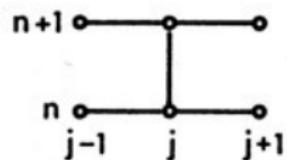
FTCS



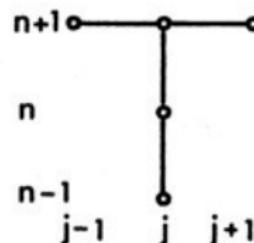
Richardson



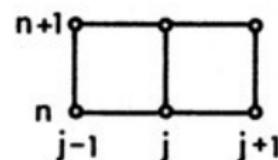
Du Fort-Frankel



Crank-Nicolson



3LFI



Linear F.E.M./  
Crank-Nicolson

# Explizite FDM - FTCS Verfahren (Skript 3.2.2/4.1)

- ▶ PDE for diffusion processes

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (17)$$

- ▶ forward time / centered space

$$\left[ \frac{\partial u}{\partial t} \right]_j^n \approx \frac{u_j^{n+1} - u_j^n}{\Delta t} \quad \left[ \frac{\partial^2 u}{\partial x^2} \right]_j^n \approx \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} \quad (18)$$

- ▶ substitute

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} = 0 \quad (19)$$

- ▶ FTCS scheme for diffusion equations

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2} (u_{j-1}^n - 2u_j^n + u_{j+1}^n), \quad Ne = \frac{\alpha \Delta t}{\Delta x^2} \quad (20)$$

Analysis of approximation schemes consists of three steps:

- ▶ Develop the **algebraic scheme**,
- ▶ Check **consistency** of the algebraic approximate equation,
- ▶ Investigate **stability** behavior of the scheme.

08.06.2018

# Eigenschaften numerischer Verfahren

Analysis of approximation schemes consists of three steps:

- ▶ Develop the **algebraic scheme**,

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2} (u_{j-1}^n - 2u_j^n + u_{j+1}^n) \quad (21)$$

- ▶ Check **consistency** of the algebraic approximate equation,

$$\lim_{\Delta t, \Delta x \rightarrow 0} |\hat{L}(u_j^n) - L(u[t_n, x_j])| = 0 \quad (22)$$

- ▶ Investigate **stability** behavior of the scheme.

$$Ne = \frac{\alpha \Delta t}{\Delta x^2} \leq 1/2 \quad (23)$$

# Lösung des FTCS Schemas

## Algebraische Schema

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2} (u_{j-1}^n - 2u_j^n + u_{j+1}^n) \quad (24)$$

## Resultierendes Gleichungssystem

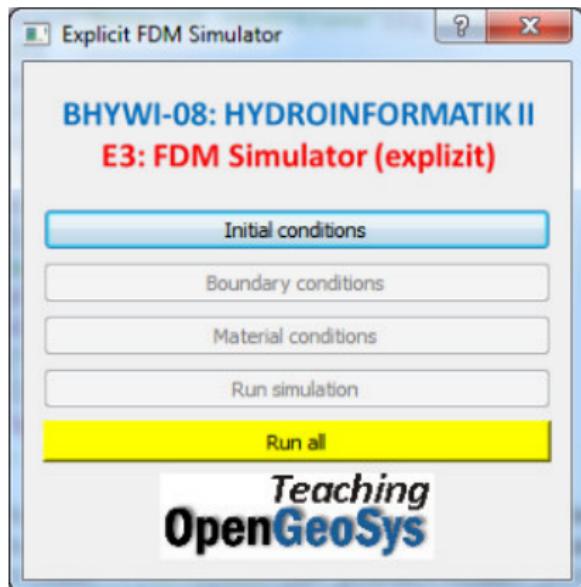
$$\mathbf{u}^{n+1} = \mathbf{A}\mathbf{u}^n \quad , \quad n = 0, 1, 2, \dots \quad (25)$$

$$\mathbf{A} = \begin{bmatrix} 1 - 2\frac{\alpha \Delta t}{\Delta x^2} & \frac{\alpha \Delta t}{\Delta x^2} & & \\ \frac{\alpha \Delta t}{\Delta x^2} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{\alpha \Delta t}{\Delta x^2} \\ & & & \frac{\alpha \Delta t}{\Delta x^2} & 1 - 2\frac{\alpha \Delta t}{\Delta x^2} \end{bmatrix}, \quad \mathbf{u}^n = \begin{bmatrix} u_2^n \\ u_3^n \\ \vdots \\ u_{np-2}^n \\ u_{np-1}^n \end{bmatrix}$$

# Übung BHYWI-08-03-E

# FDM: Explizit

## Programm-Dialog

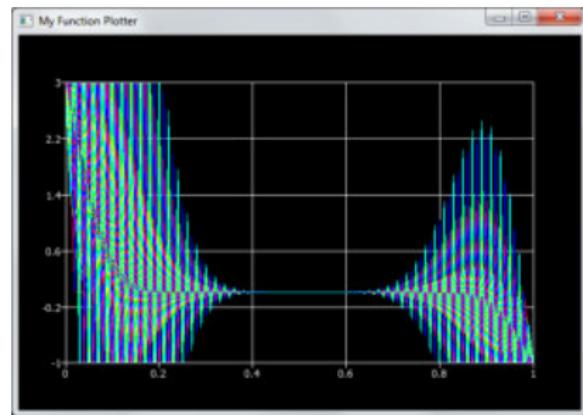
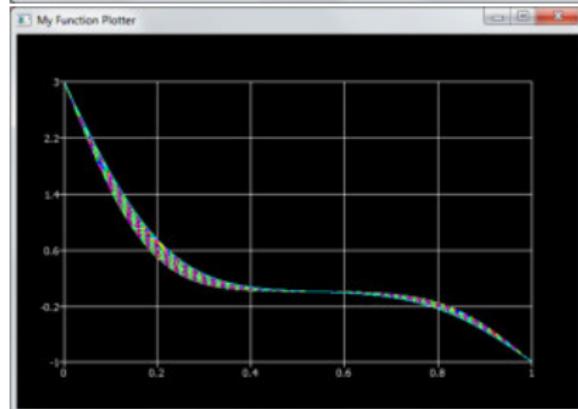
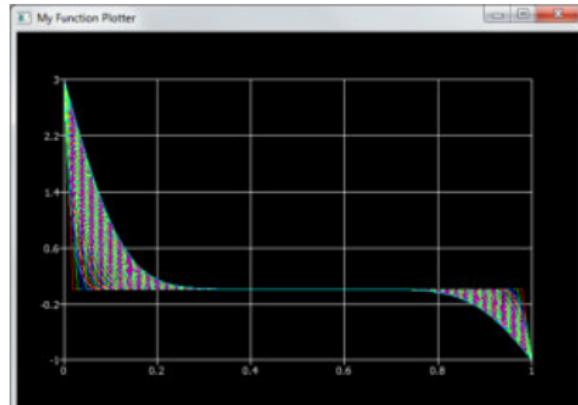


Dialog-Klasse: Konstruktor  
Dialog::Dialog

- 1 Elemente
- 2 Connects
- 3 Layout
- 4 Datenstrukturen  
(Speicherreservierung)

# FDM: Explizit

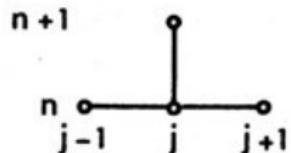
## Ergebnisse



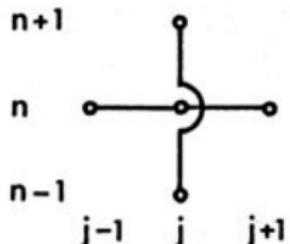
$$Ne = \frac{\alpha \Delta t}{\Delta x^2} \leq 0.5 \quad (26)$$

How sensitive ?

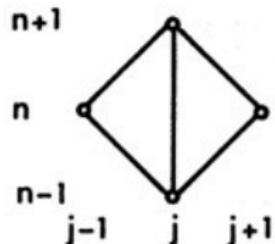
# Explizite und implizite Differenzenverfahren



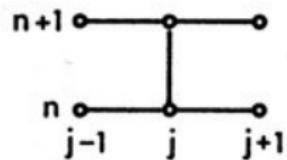
FTCS



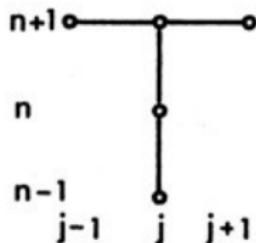
Richardson



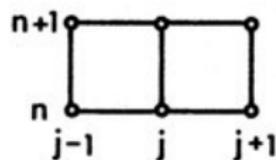
DuFort-Frankel



Crank-Nicolson



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Linear F.E.M./  
Crank-Nicolson

# Implizites Differenzenverfahren: Next Lecture

Algebraische Schema:

$$\left[ \frac{\partial^2 u}{\partial x^2} \right]_j^{n+1} \approx \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} \quad (27)$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} = 0 \quad (28)$$

$$\frac{\alpha \Delta t}{\Delta x^2} (-u_{j-1}^{n+1} + 2u_j^{n+1} - u_{j+1}^{n+1}) + u_j^{n+1} = u_j^n \quad (29)$$

# BHYWI-08: Semester-Fahrplan

## Übungen

Datum	E	Übungen
05.04.2019	00	Git und QT (Lars Bilke)
03.05.2019	01	Qt: Hallo World
10.05.2019	02	Qt: Funktionsrechner
	03	Qt: Explizite Finite-Differenzen-Methode
	04	Qt: Implizite Finite-Differenzen-Methode
	05	Qt: Gerinnehydraulik I (QAD)
	06	Qt: Gerinnehydraulik II (OOP)
	08	Qt: Gerinnehydraulik IV (interaktiv)
		...

<https://github.com/envinf/Hydroinformatik-II>

