BHYWI-08: Semester-Fahrplan 2019

Vorlesungen

Datum	V	Thema
05.04.2019	01	IT: GitHub / Qt Installation
12.04.2019	02	Grundlagen: Kontinuumsmechanik
19.04.2019	--	Ostern
26.04.2019	03	Grundlagen: Hydromechanik
03.05.2019	04	Grundlagen: Partielle Differentialgleichungen
10.05.2019	05	Grundlagen: Numerik, Qt Übung: Funktionsrechner
17.05.2019	06	Numerik: Finite Differenzen Methode I (explizit)
24.05.2019	07	Numerik: Finite Differenzen Methode II (implizit)
31.05.2019	08	Gerinnehydraulik: Theorie – Grundlagen
07.06.2019	09	Gerinnehydraulik: Programmierung, Übung
14.06.2019		Pfingsten
21.06.2019	10	Grundwassermodellierung: Catchment Übung
28.06.2019	11	Grundwassermodellierung: Datenbasierte Methoden I
05.07.2019	12	Grundwassermodellierung: Datenbasierte Methoden II
12.07.2019	13	Beleg

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Hydroinformatik II
"Prozesssimulation und Systemanalyse"
BHYWI-08-04 @ 2019
Partielle Differentialgleichungen (PDEs)

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03.05.2019 - Dresden

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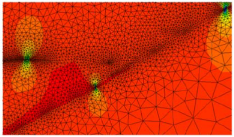
Konzept

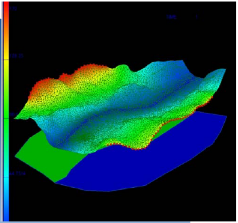
$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \mathbf{v}^E \nabla \psi$$



Basics

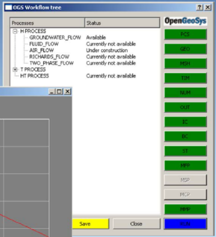
Mechanik

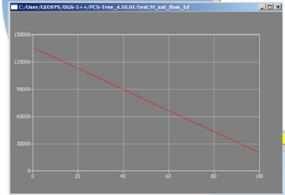
Anwendung



Numerische

Methoden





Programmierung

Visual C++

Prozessverständnis

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Inhalte

- Konzept
- Partielle Differentialgleichungen
- Klassifikation
- Beispiele
- Anfangs- und Randbedingungen

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Dimensionsanalyse

Navier-Stokes-Gleichung

Mathematical Classification (1.5)

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, ..., \frac{\partial^n \psi}{\partial x_i^n}) = 0 \qquad , \qquad i = 3 \tag{1}$$

where L is s differential operator. Second-order PDE with two independent variables are given by

$$A\frac{\partial^2 \psi}{\partial x^2} + B\frac{\partial^2 \psi}{\partial x \partial y} + C\frac{\partial^2 \psi}{\partial y^2} + D\frac{\partial \psi}{\partial x} + E\frac{\partial \psi}{\partial y} + F\psi + G = 0 \tag{2}$$

Second-order PDEs with more independent variables can be classified by examination of the eigenvalues of the matrix a_{ij}.

$$\sum_i \sum_j a_{ij} \frac{\partial \psi^2}{\partial x_i \partial x_j} + G = 0 \tag{3}$$

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Mathematical Classification (1.5)

PDE type	Discriminant	Eigenvalues	Canonical form	Example
Elliptic	$B^2 - 4AC < 0$ complex characteristics	$\forall\lambda > 0$ equal signs	$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0$	Laplace equation
Parabolic	$B^2 - 4AC = 0$	$\exists\lambda = 0$	$\frac{\partial^2\psi}{\partial\eta^2} = G$	Diffusion, Burgers equations
Hyperbolic	$B^2 - 4AC > 0$ real characteristics	$\exists\lambda < 0$ different signs	$\frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta^2} = 0$	Wave equation

General Balance Equation (1.1.7)

- Integral form

$$\int_\Omega \frac{d\psi}{dt} d\Omega = \int_\Omega \frac{\partial\psi}{\partial t} d\Omega + \int_\Omega \nabla\cdot(\mathbf{v}\psi) d\Omega - \int_\Omega \nabla\cdot(\mathbf{D}^\psi \nabla\psi) d\Omega = \int_\Omega Q^\psi d\Omega \tag{4}$$

- Differential form

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla\cdot(\mathbf{v}\psi) - \nabla\cdot(\mathbf{D}^\psi\nabla\psi) = \mathbf{Q}^\psi \tag{5}$$

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$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi}\nabla\psi) = Q^{\psi} \tag{6}$$

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial\psi}{\partial x_i}, ..., \frac{\partial^n\psi}{\partial x_i^n}) = 0 \qquad , \qquad i = 3 \tag{7}$$

where L is s differential operator.

PDE: Klassifikation

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi}\nabla\psi) = Q^{\psi} \tag{8}$$

Physical problem	Math. problem	Examples
Equilibrium problems	Elliptic equations	Irrotational incompressible flow Inviscid incompressible flow Steady state heat conduction
Propagation problems (infinite propagation speed)	Parabolic equations	Unsteady viscous flow Transient heat transfer
Propagation problems (finite propagation speed)	Hyperbolic equations	Wave propagation (vibration) Inviscid supersonic flow

- Parabolisch: Diffusion, Gerinne (nichtlinear)
- Elliptisch: Grundwasser (stationär)

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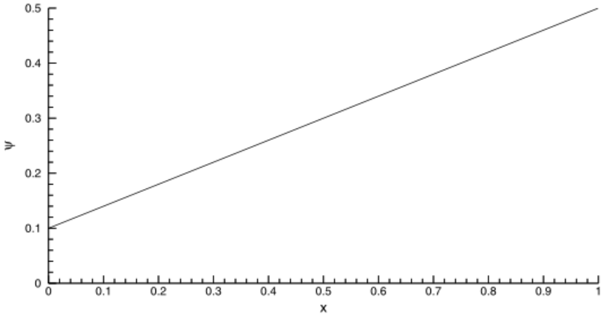
PDE: Elliptic Equation 1-D

$$\frac{d^2\psi}{dx^2} = 0 \tag{9}$$

$$\psi = ax + b \tag{10}$$



PDE: Elliptic Equation 2-D

The prototype of an elliptic equation is the Laplace equation.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{11}$$

By substitution it can be easily verified that the exact solution of the Laplace equation is

$$\psi = sin(\pi x) exp(-\pi y) \tag{12}$$

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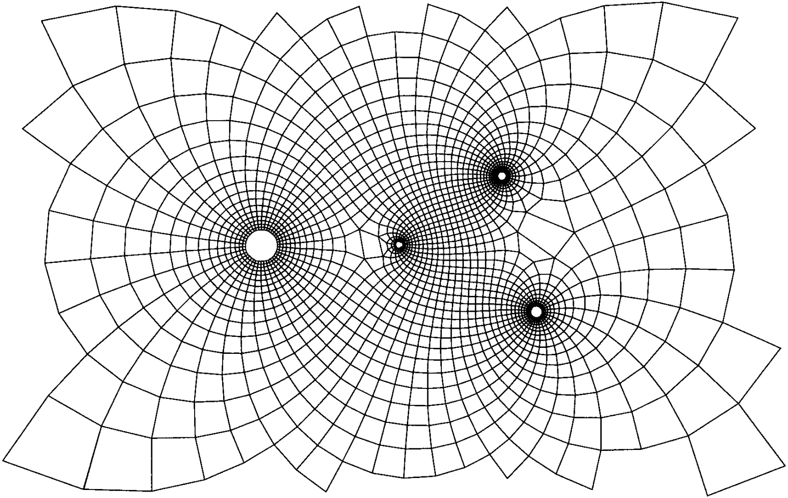
PDE: Elliptic Equation 2-D



PDE: Parabolic Equation 1-D

$$\frac{\partial \psi}{\partial t} = \alpha \frac{\partial^2 \psi}{\partial x^2} \tag{13}$$

Multiple solutions:

$$\psi(t,x) = sin(\sqrt{\pi\alpha}x)exp(-\pi t) \quad ??? \tag{14}$$

$$\psi(t,x) = sin(\frac{\pi}{\sqrt{\alpha}}x)exp(-\pi^2 t) \tag{15}$$

$$\psi(t,x) = sin(\pi x)exp(-\alpha\pi^2 t) \tag{16}$$

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PDE: Parabolic Equation 1-D

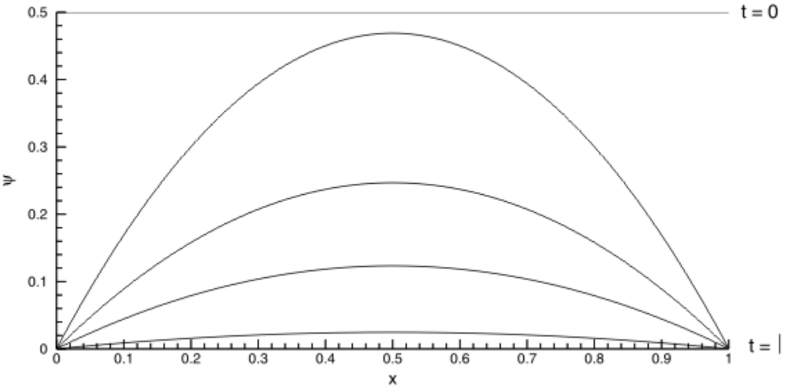

Figure: Solution of a parabolic equation

PDE: Hyperbolic Equation 1-D

$$\frac{\partial^2 \psi}{\partial^2 t} - c^2 \frac{\partial^2 \psi}{\partial^2 x} = 0 \qquad (17)$$

$$\psi(t, x) = a \cos(\frac{\pi c t}{L}) \sin(\frac{\pi x}{L}) \qquad (18)$$

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$$A\frac{\partial\psi}{\partial t} - B\frac{\partial\psi}{\partial x} = 0 \tag{19}$$

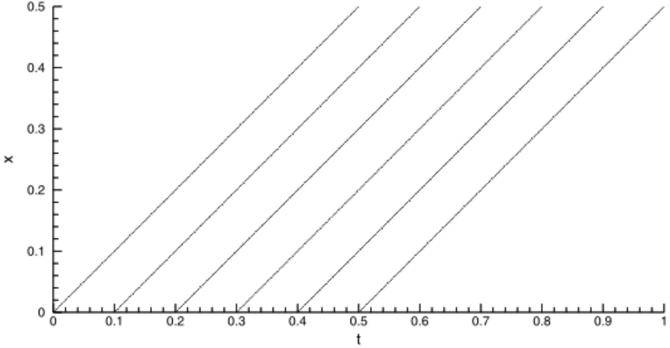

Figure: Characteristics of a hyperbolic equation

PDE: Hyperbolic Equation 1-D - Characteristics

$$A\frac{\partial \psi}{\partial t} - B\frac{\partial \psi}{\partial x} = 0 \tag{20}$$

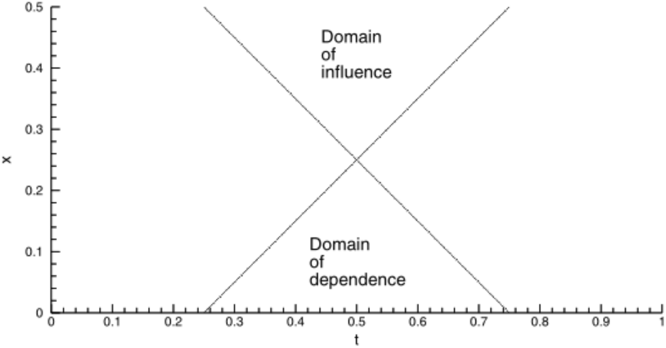

Figure: Domains of a hyperbolic equation

PDE: Equation Types

The following table gives typical examples of balance equations for the denoted quantities and their PDE types.

Physics	Equation structure	Examples
Continuity	$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$	Laplace equation
Mass/energy	$\frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} - \alpha \frac{\partial^2 \psi}{\partial x^2} = 0$	Fokker-Planck equation
Momentum	$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x}[\alpha(\psi) \frac{\partial \psi}{\partial x}] = 0$	Navier-Stokes equation

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Boundary Conditions I

The following table gives an overview on common boundary condition types and its mathematical representation.

Table: Boundary conditions types

Type of BC	Mathematical Meaning	Physical Meaning
Dirichlet type	ψ	prescribed value potential surface
Neumann type	$\nabla\psi$	prescribed flux stream surface
Cauchy type	$\psi + A\nabla\psi$	resistance between potential and stream surface

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Boundary Conditions II

To describe conditions at boundaries we can use flux expressions of conservation quantities.

Table: Fluxes through surface boundaries

Quantity	Flux term
Mass	$\rho \mathbf{v}$
Momentum	$\rho \mathbf{v}\mathbf{v} - \sigma$
Energy	$\rho e \mathbf{v} - \lambda \nabla \mathbf{T}$

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