BHYWI-08: Semester-Fahrplan 2019

Vorlesungen

Datum	V	Thema
05.04.2019	01	IT: GitHub / Qt Installation
12.04.2019	02	Grundlagen: Kontinuumsmechanik
19.04.2019	--	Ostern
26.04.2019	03	Grundlagen: Hydromechanik
03.05.2019	04	Grundlagen: Partielle Differentialgleichungen
10.05.2019	05	Grundlagen: Numerik, Qt Übung: Funktionsrechner
17.05.2019	06	Numerik: Finite Differenzen Methode I (explizit)
24.05.2019	07	Numerik: Finite Differenzen Methode II (implizit)
31.05.2019	08	Gerinnehydraulik: Theorie – Grundlagen
07.06.2019	09	Gerinnehydraulik: Programmierung, Übung
14.06.2019		Pfingsten
21.06.2019	10	Grundwassermodellierung: Catchment Übung
28.06.2019	11	Grundwassermodellierung: Datenbasierte Methoden I
05.07.2019	12	Grundwassermodellierung: Datenbasierte Methoden II
12.07.2019	13	Beleg

TECHNISCHE UNIVERSITÄT DRESDEN

Hydroinformatik II

"Prozesssimulation und Systemanalyse"

BHYWI-08-03 @ 2019

Grundlagen der Hydromechanik

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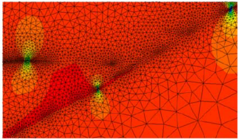
26.04.2019 - Dresden

Konzept

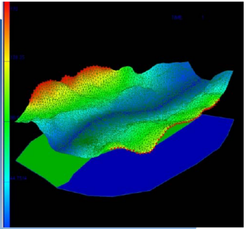
$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Basics

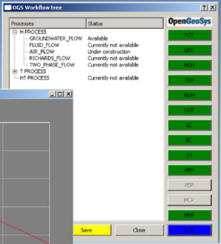
Mechanik

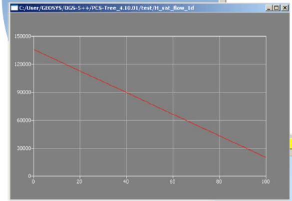
Anwendung



Numerische

Methoden





Programmierung

Visual C++

Prozessverständnis

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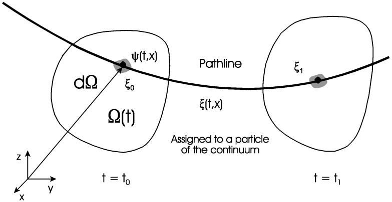
Inhalte

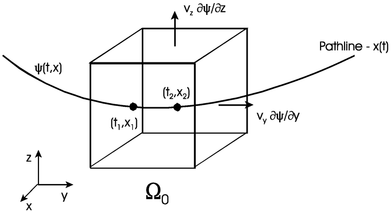
- Erhaltungsgrößen
- Massenerhaltung
- Fluidmassenerhaltung
- Diffusion
- Impulserhaltung
- Spannungen
- Fluiddruck
- Strömungsprobleme

General Balance Equation

Lagrange



Euler



General Balance Equation

$$\frac{d}{dt}\int_\Omega \psi \, d\Omega = \frac{\partial}{\partial t}\int_\Omega \psi \, d\Omega + \oint_{\partial\Omega} \mathbf{\Phi}^\psi \cdot d\mathbf{S} \quad (1)$$
$$= \frac{\partial}{\partial t}\int_\Omega \psi \, d\Omega + \int_\Omega \nabla \cdot \mathbf{\Phi}^\psi d\Omega$$

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{\Phi}^\psi \quad (2)$$
$$= \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla \psi)$$
$$= Q^\psi$$

Notation

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Conservation Quantities (1.1.2)

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \tag{3}$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ, momentum density $\rho\mathbf{v}$ or energy density e.

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
Linear momentum	$\mathbf{m}$	Linear momentum density	$\rho\mathbf{v}$
Energy	E	Energy density	$e = \rho i + \frac{1}{2}\rho v^2$

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(Phase) Mass Conservation

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = Q^{\psi} \tag{4}$$

The differential equation of mass conservation in divergence form becomes

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \tag{5}$$

Partial differentiation of the above equation gives

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0 \tag{6}$$

(Phase) Mass Conservation

Using the material (or convective) derivative the mass conservation equation can be rewritten as

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{v} \tag{7}$$

Note, above convective form of mass conservation equation becomes zero only for incompressible flows, i.e.

$$\frac{\partial \rho}{\partial t} = 0 \tag{8}$$

requires divergence-free flow.

$$\nabla \cdot \mathbf{v} = 0 \tag{9}$$

From eqn. (6) results that the above expression is the continuity equation for a homogeneous fluid ($\rho = \text{const}$).

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Conservation Quantities (1.1.2)

The amount of a quantity in a defined volume Ω is given by

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where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ, momentum density $\rho\mathbf{v}$ or energy density e.

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Momentum Conservation

$\psi = \rho \mathbf{v}$

$$\frac{\partial}{\partial t} \int\limits_{\Omega} \rho \mathbf{v} d\Omega + \oint\limits_{\partial\Omega} \mathbf{\Phi}^{m} \cdot d\mathbf{S} = \int\limits_{\Omega} \rho \mathbf{f} d\Omega \qquad (11)$$

Flux term: The advective momentum flux is defined as

$$\mathbf{\Phi}^{m} = (\rho \mathbf{v}) \otimes \mathbf{v} = (\rho \mathbf{v})\mathbf{v} \qquad (12)$$

$$\mathbf{F} = \int\limits_{\Omega} \rho \mathbf{f} d\Omega = \int\limits_{\Omega} \rho(\mathbf{f}^{e} + \mathbf{f}^{i}) d\Omega = \underbrace{\int\limits_{\Omega} \rho \mathbf{f}^{e} d\Omega}_{\text{External forces}} + \underbrace{\oint\limits_{\partial\Omega} \sigma : d\mathbf{S}}_{\text{Internal forces}} \qquad (13)$$

Substituting now flux and source terms of momentum we obtain

$$\frac{\partial}{\partial t}\int_{\Omega}\rho\mathbf{v}d\Omega+\oint_{\partial\Omega}\rho\mathbf{v}(\mathbf{v}\cdot d\mathbf{S})=\int_{\Omega}\rho\mathbf{f}^{e}d\Omega+\oint_{\partial\Omega}\sigma:d\mathbf{S} \tag{14}$$

Applying the Gauss-Ostrogradskian theorem to the surface integrals

$$\begin{aligned}\oint_{\partial\Omega}\rho\mathbf{v}(\mathbf{v}\cdot d\mathbf{S}) &= \int_{\Omega}\nabla\cdot(\rho\mathbf{v}\mathbf{v})d\Omega\\ \oint_{\partial\Omega}\sigma:d\mathbf{S} &= \int_{\Omega}\nabla\cdot\sigma\mathbf{d\Omega}\end{aligned} \tag{15}$$

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Momentum Conservation

The differential form of the momentum conservation law is then

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) = \rho\mathbf{f}^e + \nabla\cdot\sigma \tag{16}$$

The above equation is now extended by partial integration

$$\begin{aligned}\rho\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\rho}{\partial t} + (\rho\mathbf{v})\cdot\nabla\mathbf{v} + \mathbf{v}\nabla\cdot(\rho\mathbf{v}) \\ = \rho[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}] + \mathbf{v}[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})] \\ = \rho\mathbf{f}^e + \nabla\cdot\sigma\end{aligned} \tag{17}$$

Using the mass conservation equation (5) and dividing by ρ we obtain

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}^e + \frac{1}{\rho}\nabla\cdot\sigma \tag{18}$$

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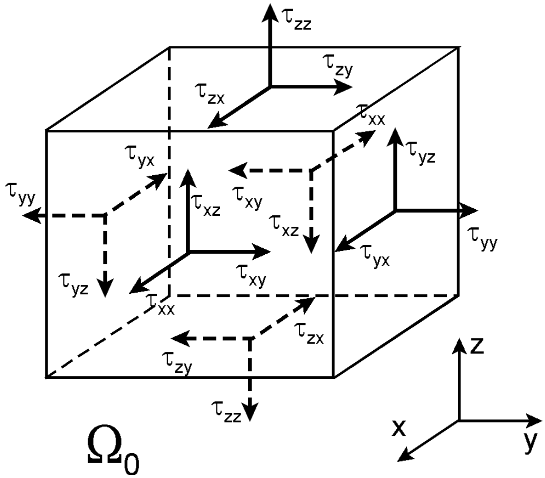
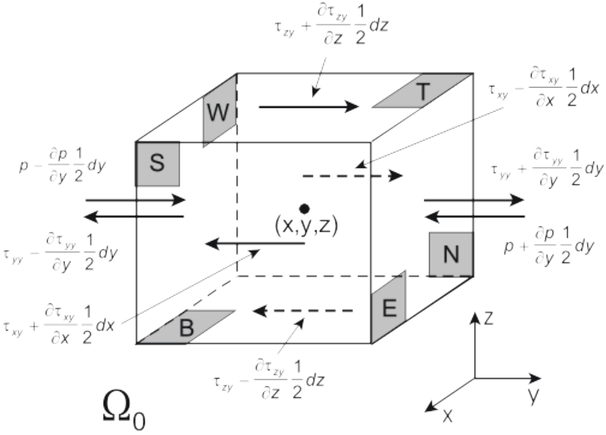
Momentum Conservation: Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \quad , \quad \boldsymbol{\tau} = \nu \nabla \mathbf{v} \tag{19}$$



Momentum Conservation: Stress Tensor

$$\boldsymbol{\tau} = \nu \nabla \mathbf{v} \tag{20}$$



Fluid Momentum Balance

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} \tag{21}$$

In index notation the above vector equation is written as

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \frac{1}{\rho}(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}) \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= \frac{1}{\rho}(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}) \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= g + \frac{1}{\rho}(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z})
\end{aligned} \tag{22}$$

with $u = v_x, v = v_y, w = v_z$ and $\mathbf{f}^e = \mathbf{g}$.

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Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \tau \tag{23}$$

Navier-Stokes Equation

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v}} \tag{24}$$

Euler Equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p \tag{25}$$

Stokes Equation

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} \tag{26}$$

Darcy Equations

$$0 = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} \tag{27}$$

(Component) Mass Conservation

$$\psi = \rho_k = C_k \tag{28}$$

$$\frac{dC_k}{dt} = \frac{\partial C_k}{\partial t} + \nabla \cdot (\mathbf{v} C_k) - \nabla \cdot (\mathbf{D_k} \nabla C_k) = Q_k \tag{29}$$