Semester-Fahrplan

Datum	V	Thema
13.04.2018	01	Einführung / Qt Installation
20.04.2018	02	Grundlagen: Kontinuumsmechanik
27.04.2018	03	Grundlagen: Hydromechanik
04.05.2018	04	Grundlagen: Partielle Differentialgleichungen
11.05.2018	05	Grundlagen: Numerik, Qt Übung: Funktionsrechner
18.05.2018	06	Numerik: Finite Differenzen Methode I (explizit)
01.06.2018	07	Numerik: Finite Differenzen Methode II (implizit)
08.06.2018	08	Gerinnehydraulik: Theorie – Grundlagen
15.06.2018	09	Gerinnehydraulik: Programmierung, Übung 1
22.06.2018	10	Gerinnehydraulik: Programmierung, Übung 2
29.06.2018	11	Grundwassermodellierung: Catchment Übung
06.07.2018	12	Grundwassermodellierung: Datenbasierte Methoden I
13.07.2018	13	Grundwassermodellierung: Datenbasierte Methoden II
20.07.2018	14	Klausurvorbereitung

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Hydroinformatik II
"Prozesssimulation und Systemanalyse"
BHYWI-08-04 @ 2018
Partielle Differentialgleichungen (PDEs)

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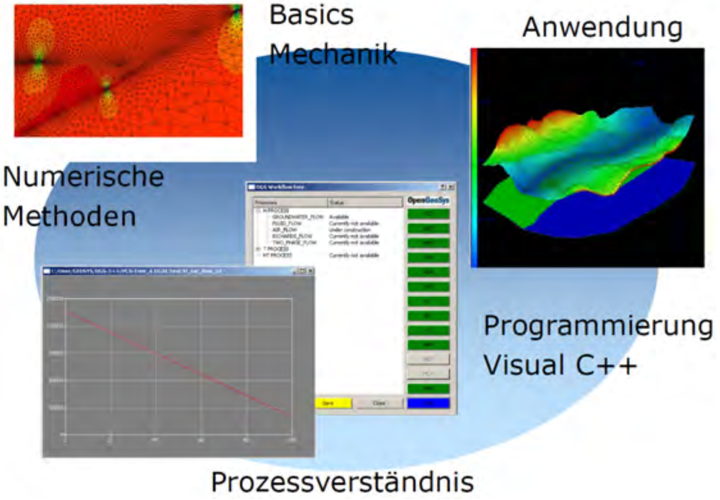
04./11.05.2018 - Dresden

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Inhalte

- Link zur letzten Vorlesung: Dimensionsanalyse

- Konzept
- Partielle Differentialgleichungen
- Klassifikation
- Beispiele
- Anfangs- und Randbedingungen

- TEXfür den Beleg

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Navier-Stokes-Gleichung

Mathematical Classification (1.5)

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, ..., \frac{\partial^n \psi}{\partial x_i^n}) = 0 \quad , \quad i = 3 \tag{1}$$

where L is s differential operator. Second-order PDE with two independent variables are given by

$$A\frac{\partial^2 \psi}{\partial x^2} + B\frac{\partial^2 \psi}{\partial x \partial y} + C\frac{\partial^2 \psi}{\partial y^2} + D\frac{\partial \psi}{\partial x} + E\frac{\partial \psi}{\partial y} + F\psi + G = 0 \tag{2}$$

Second-order PDEs with more independent variables can be classified by examination of the eigenvalues of the matrix a_{ij}.

$$\sum_i \sum_j a_{ij} \frac{\partial \psi^2}{\partial x_i \partial x_j} + G = 0 \tag{3}$$

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Mathematical Classification (1.5)

PDE type	Discriminant	Eigenvalues	Canonical form	Example
Elliptic	$B^2 - 4AC < 0$ complex characteristics	$\forall\lambda > 0$ equal signs	$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0$	Laplace equation
Parabolic	$B^2 - 4AC = 0$	$\exists\lambda = 0$	$\frac{\partial^2\psi}{\partial\eta^2} = G$	Diffusion, Burgers equations
Hyperbolic	$B^2 - 4AC > 0$ real characteristics	$\exists\lambda < 0$ different signs	$\frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta^2} = 0$	Wave equation

General Balance Equation (1.1.7)

- Integral form

$$\int_\Omega \frac{d\psi}{dt} d\Omega = \int_\Omega \frac{\partial\psi}{\partial t} d\Omega + \int_\Omega \nabla \cdot (\mathbf{v}\psi) d\Omega - \int_\Omega \nabla \cdot (\mathbf{D}^\psi \nabla\psi) d\Omega = \int_\Omega Q^\psi d\Omega \quad (4)$$

- Differential form

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla\psi) = \mathbf{Q}^\psi \quad (5)$$

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PDE: Definition

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi}\nabla\psi) = \mathbf{Q}^{\psi} \tag{6}$$

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, ..., \frac{\partial^n \psi}{\partial x_i^n}) = 0 \qquad , \qquad i = 3 \tag{7}$$

where L is s differential operator.

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PDE: Klassifikation

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = \mathbf{Q}^{\psi} \qquad (8)$$

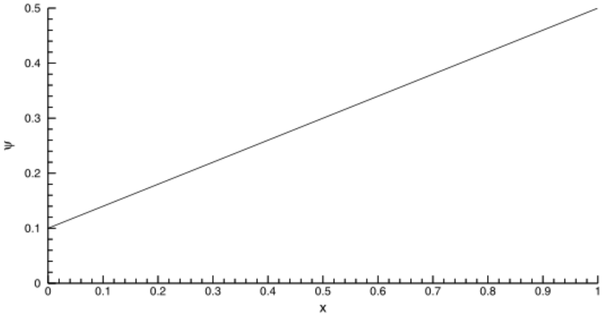
Physical problem	Math. problem	Examples
Equilibrium problems	Elliptic equations	Irrotational incompressible flow Inviscid incompressible flow Steady state heat conduction
Propagation problems (infinite propagation speed)	Parabolic equations	Unsteady viscous flow Transient heat transfer
Propagation problems (finite propagation speed)	Hyperbolic equations	Wave propagation (vibration) Inviscid supersonic flow

- Parabolisch: Diffusion, Gerinne (nichtlinear)
- Elliptisch: Grundwasser (stationär)

PDE: Elliptic Equation 1-D

$$\frac{d^2\psi}{dx^2} = 0 \quad (9)$$

$$\psi = ax + b \quad (10)$$



PDE: Elliptic Equation 2-D

The prototype of an elliptic equation is the Laplace equation.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{11}$$

By substitution it can be easily verified that the exact solution of the Laplace equation is

$$\psi = sin(\pi x)exp(-\pi y) \tag{12}$$

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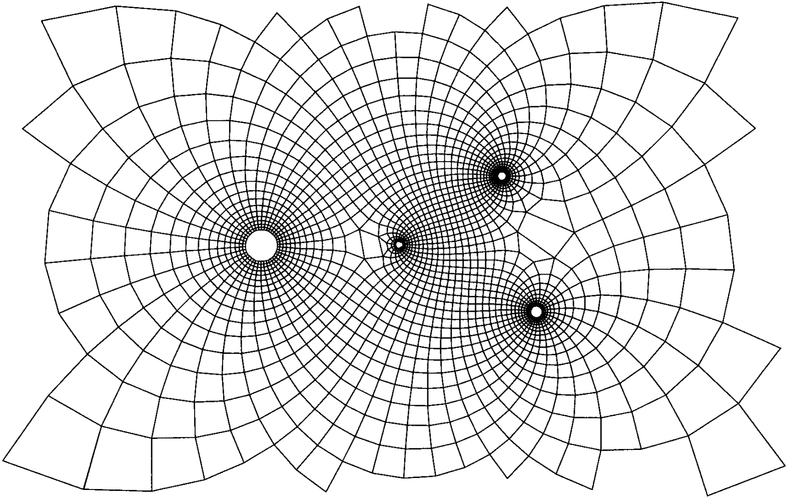
PDE: Elliptic Equation 2-D



PDE: Parabolic Equation 1-D

$$\frac{\partial \psi}{\partial t} = \alpha \frac{\partial^2 \psi}{\partial x^2} \tag{13}$$

Multiple solutions:

$$\psi(t, x) = sin(\sqrt{\pi\alpha}x)exp(-\pi t) \quad ??? \tag{14}$$

$$\psi(t, x) = sin(\frac{\pi}{\sqrt{\alpha}}x)exp(-\pi^2 t) \tag{15}$$

$$\psi(t, x) = sin(\pi x)exp(-\alpha\pi^2 t) \tag{16}$$

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PDE: Parabolic Equation 1-D

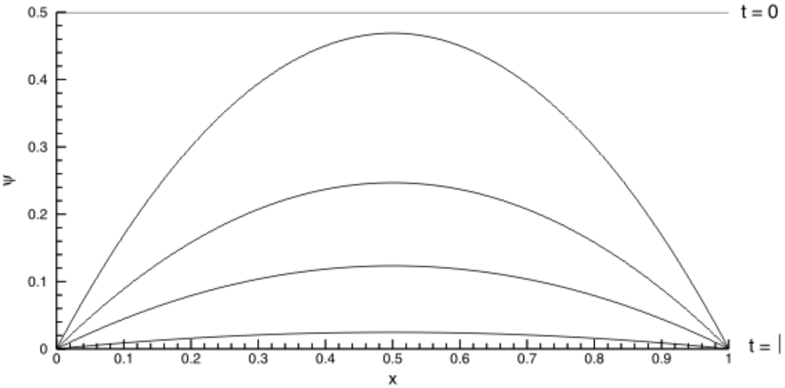

Figure: Solution of a parabolic equation

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PDE: Hyperbolic Equation 1-D

$$\frac{\partial^2 \psi}{\partial^2 t} - c^2 \frac{\partial^2 \psi}{\partial^2 x} = 0 \tag{17}$$

$$\psi(t, x) = a \cos(\frac{\pi c t}{L}) \sin(\frac{\pi x}{L}) \tag{18}$$

$$A\frac{\partial\psi}{\partial t} - B\frac{\partial\psi}{\partial x} = 0 \tag{19}$$

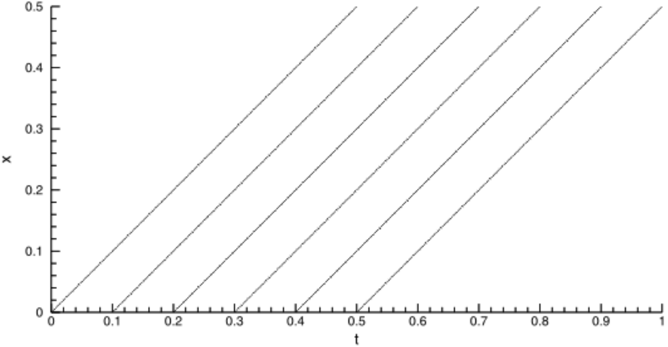

Figure: Characteristics of a hyperbolic equation

PDE: Hyperbolic Equation 1-D - Characteristics

$$A\frac{\partial\psi}{\partial t} - B\frac{\partial\psi}{\partial x} = 0 \qquad (20)$$

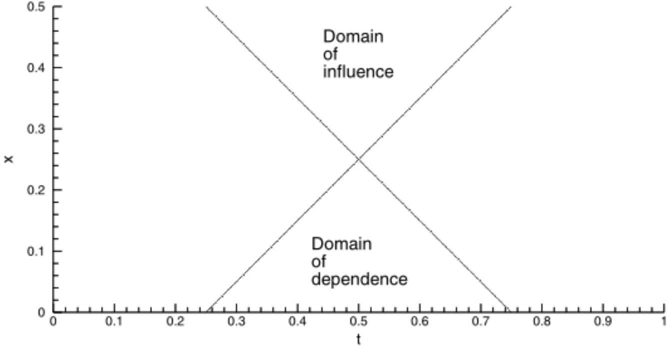

Figure: Domains of a hyperbolic equation

PDE: Equation Types

The following table gives typical examples of balance equations for the denoted quantities and their PDE types.

Physical meaning	Equation structure	Examples
Continuity	$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$	Laplace equation
Mass/energy	$\frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} - \alpha \frac{\partial^2 \psi}{\partial x^2} = 0$	Fokker-Planck equation
Momentum	$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x}[\alpha(\psi) \frac{\partial \psi}{\partial x}] = 0$	Navier-Stokes equation

Boundary Conditions I

The following table gives an overview on common boundary condition types and its mathematical representation.

Table: Boundary conditions types

Type of BC	Mathematical Meaning	Physical Meaning
Dirichlet type	ψ	prescribed value potential surface
Neumann type	$\nabla\psi$	prescribed flux stream surface
Cauchy type	$\psi + A\nabla\psi$	resistance between potential and stream surface

Boundary Conditions II

To describe conditions at boundaries we can use flux expressions of conservation quantities.

Table: Fluxes through surface boundaries

Quantity	Flux term
Mass	$\rho \mathbf{v}$
Momentum	$\rho \mathbf{v}\mathbf{v} - \sigma$
Energy	$\rho e \mathbf{v} - \lambda \nabla \mathbf{T}$

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