Derivation of a solution for general initial and boundary conditions for a leaky strip aquifer obeying the Dupuit assumptions.

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As demonstrated in the appendix, the equation in dimensionless space coordinate x and time coordinate t is of the form:

$$\frac{\partial H}{\partial t} = \frac{1}{\mu}\frac{\partial^2 H}{\partial x^2} + AH + B(t) \tag{1}$$

where A is given by

$$A = \frac{aL^2}{\mu KD} \tag{2}$$

and $B(t)$ represents head-independent fluxes (rainfall and the head-independent component of the flow to/from a deeper aquifer:

$$B(t) = \frac{L^2}{KD}\frac{b + R(t)}{\mu} \tag{3}$$

The IC and BC are:

$$H(x,0) \text{ arbitrary} \tag{4}$$

$$\frac{\partial H(0,t)}{\partial x} = 0 \tag{5}$$

$$H(1,t) \text{ arbitrary} \tag{6}$$

Equation (1) is a parabolic, non-homogeneous 2nd order PDE with non-homogeneous BCs. The following substitution removes the term *AH* (see also Farlow, 1993, pp. 58-61):

$$H(x,t) = \mathrm{e}^{At} W(x,t) - \frac{B(t)}{A} \tag{7}$$

The system becomes:

$$\frac{\partial W}{\partial t} = \frac{1}{\mu}\frac{\partial^2 W}{\partial x^2} + \frac{\mathrm{e}^{-At}}{A}\frac{\mathrm{d}B}{\mathrm{d}t} \tag{8}$$

$$W(x,0) = H(x,0) + \frac{B(0)}{A} \tag{9}$$

$$\frac{\partial W(0,t)}{\partial x} = 0 \tag{10}$$

$$W(1,t) = \mathrm{e}^{-At}\left(H(1,t) + \frac{B(t)}{A}\right) \tag{11}$$

The non−homogeneous term in the PDE is now independent of *H*. The BC at $x = 1$ is non-homogeneous. We seek a substitution that makes both BCs homogeneous (compare Farlow, 1993, p. 43−47). Thus:

$$W(x,t) = \varsigma(t)(1-x) + \beta(t)x + V(x,t) \tag{12}$$

where $V(x,t)$ is the new dependent variable and $\zeta(t)$ and $\beta(t)$ are as yet unknown functions. From the first BC (Eq. (10)) we have:

$$\frac{\partial W(0,t)}{\partial x} = -\varsigma(t) + \beta(t) + \frac{\partial V(0,t)}{\partial x} = 0 \tag{13}$$

From the second BC follows:

$$W(1,t) = \beta(t) + V(1,t) = \mathrm{e}^{-At}\left(H(1,t) + \frac{B(t)}{A} \right) \tag{14}$$

To ensure homogeneous BCs for $V(x,t)$, $\beta(t)$ needs to be defined as:

$$\beta(t) = \mathrm{e}^{-At}\left(H(1,t) + \frac{B(t)}{A} \right) \tag{15}$$

From Eqs. (13) and (15) follows the definition of $\zeta(t)$:

$$\varsigma(t) = \mathrm{e}^{-At}\left(H(1,t) + \frac{B(t)}{A} \right) \tag{16}$$

The substitution in Eq. (12) thus becomes:

$$W(x,t) = \mathrm{e}^{-At}\left(H(1,t) + \frac{B(t)}{A} \right) + V(x,t) \tag{17}$$

The system now becomes:

$$\frac{\partial V}{\partial t} = \frac{1}{\mu}\frac{\partial^2 V}{\partial x^2} + \mathrm{e}^{-At}\left(AH(1,t) + B(t) - \frac{\mathrm{d}H(1,t)}{\mathrm{d}t} \right) \tag{18}$$

(Note that the term with $\mathrm{d}B/\mathrm{d}t$ in Eq. (8) is cancelled by a similar term that arises when Eq. (17) is differentiated with respect to time.)

$$V(x,0) = H(x,0) - H(1,0) \tag{19}$$

$$\frac{\partial V(0,t)}{\partial x} = 0 \tag{20}$$

$$V(1,t) = 0 \tag{21}$$

This system can be solved by the method of eigenfunction expansions (Farlow, 1993, p. 64−70). We start from the solution of the associated homogeneous problem by separation of variables:

$$V(x,t) = X(x)T(t) \tag{22}$$

where X and T are as yet unknown functions. From this follows (Farlow, 1993, p. 33−41):

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \lambda^2 X = 0 \tag{23}$$

The BCs for V give for X:

$$\frac{\mathrm{d}X(0)}{\mathrm{d}x} = 0 \tag{24}$$

$$X(1) = 0 \tag{25}$$

The general solution for X is:

$$X(x) = C\sin(\lambda x) + E\cos(\lambda x) \tag{26}$$

with C and E unknown constants. From Eq. (24) follows that C = 0. From Eq. (25) then follows that cos(λx) = 0. Thus, we have:

$$X_n(x) = E_n \cos(\lambda_n x),\ \lambda_n = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots, n = 0,1,2,\ldots \tag{27}$$

Equation (B25) gives a valid solution for arbitrary values of E_n. These are therefore set to 1. The non−homogeneous term in Eq. (18) needs to be expressed as a series of X_n:

$$\mathrm{e}^{-At}\left(AH(1,t) + B(t) - \frac{\mathrm{d}H(1,t)}{\mathrm{d}t} \right) = \sum_{n=0}^{\infty} f_n(t) X_n(x) \tag{28}$$

with $f_n(t)$ determined from

$$\int_0^1 \mathrm{e}^{-At}\left(AH(1,t)+B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right)X_m\mathrm{d}x=\sum_{n=0}^{\infty} f_n(t)\int_0^1 X_m X_n \mathrm{d}x \qquad (29)$$

Note that X_m and X_n are orthogonal on the integration interval for $m \neq n$ (see Bruggeman, 1999, p. 701−703), and the integral of their product therefore equals 0. In the sum, only the term for $m = n$ is non−zero:

$$\mathrm{e}^{-At}\left(AH(1,t)+B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right)\int_0^1 \cos(\lambda_m x)\mathrm{d}x = f_m(t)\int_0^1 \cos^2(\lambda_m x)\mathrm{d}x \qquad (30)$$

Note that setting the value of E_m equal to one above is accommodated by the freedom to determine f_m. Evaluating the integrals leads to:

$$\begin{aligned}
&\mathrm{e}^{-At}\left(AH(1,t)+B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right)\left[\frac{\sin\left[\left(m+\frac{1}{2}\right)\pi x\right]}{\left(m+\frac{1}{2}\right)\pi}\right]_0^1 \\
&= f_m(t)\left[\frac{(2m+1)\pi x+\sin[(2m+1)\pi x]}{(4m+2)\pi}\right]_0^1 \\
&\Leftrightarrow \mathrm{e}^{-At}\left(AH(1,t)+B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right)\frac{(-1)^m}{\left(m+\frac{1}{2}\right)\pi}=\frac{1}{2}f_m(t) \\
&\Leftrightarrow f_m(t)=\frac{2}{\pi}\mathrm{e}^{-At}\left(AH(1,t)+B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right)\frac{(-1)^m}{\left(m+\frac{1}{2}\right)}
\end{aligned} \qquad (31)$$

With this, the series expansion of the non−homogeneous term is:

$$\mathrm{e}^{-At}\left(AH(1,t)+B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right)$$
$$=\frac{2}{\pi}\mathrm{e}^{-At}\left(AH(1,t)+B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\cos\left[\left(n+\frac{1}{2}\right)\pi x\right] \tag{32}$$

The solution $V(x,t)$ was assumed of the form:

$$V(x,t)=X(x)T(t) \tag{22}$$

The linearity of the system of Eqs. (18)−(21) and the homogeneity of the BCs ensures that any linear combination of solutions is also a solution:

$$V(x,t)=\sum_{n=0}^{\infty}X_n(x)T_n(t) \tag{33}$$

T_n needs to be chosen such that the following equality holds for an individual solution V_n:

$$\frac{\partial V}{\partial t}=\frac{1}{\mu}\frac{\partial^2 V}{\partial x^2}+\frac{2}{\pi}\mathrm{e}^{-At}\left(AH(1,t)+B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right)\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}X_n \tag{34}$$

Including in Eq. (34) the SOV solution to V to gives:

$$X_n\frac{\mathrm{d}T_n}{\mathrm{d}t}=\frac{T_n}{\mu}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}+\frac{2}{\pi}\mathrm{e}^{-At}\left(AH(1,t)+B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right)\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}X_n \tag{35}$$

We replace the second spatial derivative of X_n by $-\lambda_n^2 X_n$ (Eq. (23)). This allows X_n to be divided out:

$$\frac{\mathrm{d}T_n}{\mathrm{d}t} = -\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 T_n + \frac{2}{\pi}\mathrm{e}^{-At}\left(AH(1,t)+B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right)\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}$$

$$= -\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 T_n + \frac{2A}{\pi}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\mathrm{e}^{-At}H(1,t) \tag{36}$$

$$+\frac{2}{\pi}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\mathrm{e}^{-At}B(t) - \frac{2}{\pi}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\mathrm{e}^{-At}\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}$$

This is an ordinary differential equation in T_n. To simplify the notation we introduce

$$\gamma_n = -\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 \tag{37}$$

and

$$\eta_n = \frac{2}{\pi}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)} \tag{38}$$

to simplify Eq. (36) to:

$$\frac{\mathrm{d}T_n}{\mathrm{d}t} = \gamma_n T_n + \eta_n AH(1,t)\mathrm{e}^{-At} + \eta_n \mathrm{e}^{-At}\left(B(t)-\frac{\mathrm{d}H(1,t)}{\mathrm{d}t}\right) \tag{39}$$

The solution to Eq. (39) is:

$$T_n(t) = F_n \mathrm{e}^{\gamma_n t} + \int_0^t \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\eta_n\left(AH(1,\tau)+B(\tau)-\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau \tag{40}$$

with τ [T] an integration variable, and F_n a constant that needs to be determined from the IC.

At t = 0, the integral becomes zero:

$$T_n(0) = F_n \tag{41}$$

To find F_n we need to expand the IC of $V(x,t)$ (Eq. (19)) in a series of the eigenfunctions $X_n(x)$.

$$F_n = T_n(0) = \frac{\int_0^1 \left(H(x,0) - H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x}{\int_0^1 \cos^2\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x} = 2\int_0^1 \left(H(x,0) - H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x \tag{42}$$

The full solution for $T_n(t)$ thus becomes:

$$T_n(t) = 2\mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 t}\int_0^1 \left(H(x,0) - H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x + \frac{2}{\pi}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\int_0^t \mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 (t-\tau) - A\tau}\left(AH(1,\tau) + B(\tau) - \frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau \tag{43}$$

The solution for $V(x,t)$ then is:

$$V(x,t) = \sum_{n=0}^{\infty} X_n(x)T_n(t) = \frac{2}{\pi}\sum_{n=0}^{\infty}\left\{\begin{aligned}&\pi\mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 t}\int_0^1 \left(H(x,0) - H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x \\ &+ \frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\int_0^t \mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 (t-\tau) - A\tau}\left(AH(1,\tau) + B(\tau) - \frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau\end{aligned}\right\}\cos\left[\left(n+\frac{1}{2}\right)\pi x\right] \tag{44}$$

With Eq. (17) we find for $W(x,t)$:

$$W(x,t) = \mathrm{e}^{-At}\left(H(1,t)+\frac{B(t)}{A}\right)+V(x,t)$$
$$= \mathrm{e}^{-At}\left(H(1,t)+\frac{B(t)}{A}\right)$$
$$+\frac{2}{\pi}\sum_{n=0}^{\infty}\left\{\begin{array}{l}\pi \mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 t}\int_0^1\big(H(x,0)-H(1,0)\big)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x \\ +\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\int_0^t \mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 (t-\tau)-A\tau}\left(AH(1,\tau)+B(\tau)-\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau\end{array}\right\}\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]$$

(45)

And finally, with Eq. (7) we obtain the solution for $H(x,t)$:

$$H(x,t) = \mathrm{e}^{At}W(x,t)-\frac{B(t)}{A}$$
$$= H(1,t)$$
$$+\frac{2}{\pi}\sum_{n=0}^{\infty}\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\left(\begin{array}{l}\pi \mathrm{e}^{\left\{A-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2\right\}t}\int_0^1\big(H(x,0)-H(1,0)\big)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x \\ +\mathrm{e}^{At}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}\int_0^t \mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 (t-\tau)-A\tau}\left(AH(1,\tau)+B(\tau)-\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau\end{array}\right)$$

(46)

With Eqs. (2) and (3) this leads to

$$
\begin{aligned}
&H(x,t) = H(1,t) \\
&+\frac{2}{\pi}\sum_{n=0}^{\infty}\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]
\left(\begin{aligned}
&\pi \mathrm{e}^{\left\{\frac{aL^2}{KD}-\left[\left(n+\frac{1}{2}\right)\pi\right]^2\right\}\frac{t}{\mu}} \\
&\int_0^1 \left(H(x,0)-H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x \\
&+\mathrm{e}^{\frac{aL^2}{\mu KD}t}\ \frac{(-1)^n}{\left(n+\frac{1}{2}\right)} \\
&\int_0^t \mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2(t-\tau)-\frac{aL^2}{\mu KD}\tau}\left(\frac{aL^2}{\mu KD}H(1,\tau)+\frac{L^2}{KD}\frac{b+R(\tau)}{\mu}-\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau
\end{aligned}\right)
\end{aligned}
\tag{47}
$$

The flux per unit length of surface water bank (parallel to coordinate x_2) is then

$$
\begin{aligned}
&\frac{Q}{\Delta x_2}(1,t) = -KD\left.\frac{\partial H}{\partial x_1}\right|_{x_1=L} = -\frac{KD}{L}\left.\frac{\partial H}{\partial x}\right|_{x=1} \\
&= \frac{2KD}{L}\sum_{n=0}^{\infty}
\left(\begin{aligned}
&(-1)^n \pi\left(\mathrm{n}+\frac{1}{2}\right)\mathrm{e}^{\left\{\frac{aL^2}{KD}-\left[\left(n+\frac{1}{2}\right)\pi\right]^2\right\}\frac{t}{\mu}} \\
&\int_0^1 \left(H(x,0)-H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x \\
&+\mathrm{e}^{\frac{aL^2}{\mu KD}t} \\
&\int_0^t \mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2(t-\tau)-\frac{aL^2}{\mu KD}\tau}\left(\frac{aL^2}{\mu KD}H(1,\tau)+\frac{L^2}{KD}\frac{b+R(\tau)}{\mu}-\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau
\end{aligned}\right)
\end{aligned}
\tag{48}
$$

The average hydraulic head is

$$\overline{H}(t)=\int_0^1 H(x,t)\mathrm{d}x$$
$$=H(1,t)$$
$$+\frac{2}{\pi^2}\sum_{n=0}^{\infty}\left(\begin{array}{l}\dfrac{(-1)^n}{\left(n+\frac{1}{2}\right)}\pi e^{\left\{\frac{aL^2}{KD}-\left[\left(n+\frac{1}{2}\right)\pi\right]^2\right\}\frac{t}{\mu}}\\ \int_0^1\left(H(x,0)-H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x\\ +e^{\frac{aL^2}{\mu KD}t}\left(n+\frac{1}{2}\right)^{-2}\\ \int_0^t e^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2(t-\tau)-\frac{aL^2}{\mu KD}\tau}\left(\frac{aL^2}{\mu KD}H(1,\tau)+\frac{L^2}{KD}\frac{b+R(\tau)}{\mu}-\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau\end{array}\right) \tag{49}$$

By introducing

$$M_n(t)=\pi e^{\left\{\frac{aL^2}{KD}-\left[\left(n+\frac{1}{2}\right)\pi\right]^2\right\}\frac{t}{\mu}}\int_0^1\left(H(x,0)-H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x$$
$$+\frac{(-1)^n}{n+\frac{1}{2}}e^{\frac{aL^2}{\mu KD}t}\int_0^t e^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2(t-\tau)-\frac{aL^2}{\mu KD}\tau}\left(\frac{aL^2}{\mu KD}H(1,\tau)+\frac{L^2}{KD}\frac{b+R(\tau)}{\mu}-\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau \tag{50}$$

the expressions above simplify to

$$H(x,t)=H(1,t)+\frac{2}{\pi}\sum_{n=0}^{\infty}\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]M_n(t) \tag{47a}$$

$$\frac{Q}{\Delta x_2}(1,t)=\frac{2KD}{L}\sum_{n=0}^{\infty}\frac{\left(n+\frac{1}{2}\right)}{(-1)^n}M_n(t) \tag{48a}$$

$$\overline{H}(t)=H(1,t)+\frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)}M_n(t) \tag{49a}$$

The upscaled conductivity that arises is similar to that defined by de Rooij (2012) (Eq. (56):

$$K_{\mathrm{up}}(t) = \frac{Q}{\Delta x_2\left(\overline{H}(t) - H(1,t)\right)} = \frac{\pi^2 KD}{L} \frac{\sum_{n=0}^{\infty} \frac{\left(n+\frac{1}{2}\right)}{(-1)^n} M_n(t)}{\sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{1}{2}\right)} M_n(t)} \tag{51}$$

Equations (47)−(50), and therefore Eqs. (47a)−(49a) and (51), contain two integrals:

$$\int_0^1 \left(H(x,0) - H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x \tag{52}$$

and

$$\int_0^t \mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2 (t-\tau) - \frac{aL^2}{\mu KD}\tau} \left(\frac{aL^2}{\mu KD} H(1,\tau) + \frac{L^2}{KD}\frac{b+R(\tau)}{\mu} - \frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau \tag{53}$$

The problem analysis benefits considerably if these integrals can be evaluated analytically. For Equation (52) this is feasible for two hydrologically relevant cases: that of a horizontal groundwater level at $t = 0$, and that for steady flow in a non-leaky aquifer during constant and uniform groundwater recharge. For an initially horizontal groundwater level $H(x,0) = H(0)$, Eq. (52) results in:

$$\int_0^1 \left(H(0) - H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x = \frac{(-1)^n}{\left(n+\frac{1}{2}\right)\pi}\left(H(0) - H(1,0)\right) \tag{54}$$

For steady recharge, the groundwater level is parabolic. The expression for $H(x,0) - H(1,0)$ becomes

$$H(x,0) - H(1,0) = \frac{RL^2}{2KD}\left(1 - x^2\right) \tag{55}$$

For this case, the integral in Eq. (52) is

$$\int_0^1 \left(H(x,0) - H(1,0)\right)\cos\left[\left(n + \frac{1}{2}\right)\pi x\right]\mathrm{d}x = \frac{RL^2}{2KD}\frac{(-1)^n}{(n+1)\pi}\left\{1 - \frac{[(n+1)\pi]^2 - 2}{[(n+1)\pi]^2}\right\} \tag{56}$$

For the case of arbitrary $H(x,0)$, it should be noted that Eq. (52) has an integrand consisting of a product of a term that will often slowly vary over the integration interval with an oscillatory term that will oscillate at very high frequency for large n. Integration of such functions can be difficult (see Mikš et al., 2010). Since the slow−changing part can be approximated by piecewise linear segments between observation/simulation points, and the oscillating function is straightforward (constant wavelength and amplitude over the integration interval), analytical integration is feasible. Let x_i denote one of k observation points, with $x_1 = 0$ and $x_k = 1$. The integral of Eq. (52) then becomes:

$$\int_0^1 \left(H(x,0)-H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x$$
$$=\sum_{i=1}^{k-1}\int_{x_i}^{x_{i+1}}\left[H(x_i,0)+\frac{H(x_{i+1},0)-H(x_i,0)}{x_{i+1}-x_i}\left(x-x_i\right)-H(1,0)\right]\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x$$
$$=-H(1,0)\int_0^1\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x+\sum_{i=1}^{k-1}\left(H(x_i,0)-\frac{H(x_{i+1},0)-H(x_i,0)}{x_{i+1}-x_i}x_i\right)\int_{x_i}^{x_{i+1}}\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x$$
$$+\sum_{i=1}^{k-1}\frac{H(x_{i+1},0)-H(x_i,0)}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}x\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x$$
$$=-\frac{H(1,0)}{\left(n+\frac{1}{2}\right)\pi}\left(-1\right)^n$$
$$+\frac{1}{\left(n+\frac{1}{2}\right)\pi}\sum_{i=1}^{k-1}\left(H(x_i,0)-\frac{H(x_{i+1},0)-H(x_i,0)}{x_{i+1}-x_i}x_i\right)\left\{\sin\left[\left(n+\frac{1}{2}\right)\pi x_{i+1}\right]-\sin\left[\left(n+\frac{1}{2}\right)\pi x_i\right]\right\}$$
$$+\frac{1}{\left(n+\frac{1}{2}\right)\pi}\sum_{i=1}^{k-1}\frac{H(x_{i+1},0)-H(x_i,0)}{x_{i+1}-x_i}\left(\begin{array}{l}\left\{x_{i+1}\sin\left[\left(n+\frac{1}{2}\right)\pi x_{i+1}\right]-x_i\sin\left[\left(n+\frac{1}{2}\right)\pi x_i\right]\right\}\\ +\frac{1}{\left(n+\frac{1}{2}\right)\pi}\left\{\cos\left[\left(n+\frac{1}{2}\right)\pi x_{i+1}\right]-\cos\left[\left(n+\frac{1}{2}\right)\pi x_i\right]\right\}\end{array}\right)$$
(57)

For the integral in Eq. (53) it is useful to note that rainfall and (potential) evapotranspiration are often recorded as daily totals, and thus can be readily converted to average daily flux densities. De Rooij (2012) showed that this resolution is adequate for regular groundwater modeling purposes. If groundwater recharge is assumed to be equal to the net infiltration, $R(t)$ then becomes a staircase function.

The surface water level is usually measured at distinct times. Linear interpolation between these times makes $H(1,t)$ a continuous, piecewise linear function and its temporal derivative a staircase function. The integral is analytically tractable for both the piecewise linear and the staircase functions. The time line needs to be divided into intervals defined by the rainfall and surface water recording times: irrespective of the variable being measured, a new time interval is started for all terms in the integral at each data collection time.

In Eq. (53), the time for which the integral is sought is denoted by t, and τ denotes the dimensionless time running between zero and t. in the following, $R(\tau)$ and $H(1,\tau)$ are assumed to have been recorded simultaneously at $\tau = 0$ and at an additional j values of τ (denoted τ_i) smaller than t. At time t, $R_{j+1}(t)$, $H(1,t)$, and $\mathrm{d}H(1,\tau)/\mathrm{d}\tau|_t$ either have to be observed, or interpolated between observations. We invoke Eqs. (2) and (37) to simplify the notation of Eq. (53), and separate the integration intervals:

$$\begin{aligned}&\int_0^{\tau_1} \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\left[AH(1,\tau)+\frac{L^2}{\mu KD}\big(b+R(\tau)\big)-\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right]\mathrm{d}\tau\\&+\sum_{i=1}^{j}\int_{\tau_i}^{\tau_{i+1}} \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\left[AH(1,\tau)+\frac{L^2}{\mu KD}\big(b+R(\tau)\big)-\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right]\mathrm{d}\tau\end{aligned} \qquad (58)$$

On each interval, $H(1,\tau)$ is linear, and $R(\tau)$ and $\mathrm{d}H(1,\tau)/\mathrm{d}\tau$ are constant. The various terms in the integrand are separated accordingly to allow them to be evaluated individually.

$$\begin{aligned}&A\int_0^{\tau_1} \mathrm{e}^{\gamma_n(t-\tau)-A\tau}H(1,\tau)\mathrm{d}\tau\\&+\left[\frac{L^2}{\mu KD}\big(b+R_1\big)-\left.\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right|_1\right]\int_0^{\tau_1}\mathrm{e}^{\gamma_n(t-\tau)-A\tau}\mathrm{d}\tau\\&+A\sum_{i=1}^{j}\int_{\tau_i}^{\tau_{i+1}} \mathrm{e}^{\gamma_n(t-\tau)-A\tau}H(1,\tau)\mathrm{d}\tau\\&+\sum_{i=1}^{j}\left[\frac{L^2}{\mu KD}\big(b+R_{i+1}\big)-\left.\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right|_{i+1}\right]\int_{\tau_i}^{\tau_{i+1}}\mathrm{e}^{\gamma_n(t-\tau)-A\tau}\mathrm{d}\tau\end{aligned} \qquad (59)$$

Here, it is assumed that the constant values of R and $\mathrm{d}H(1,\tau)/\mathrm{d}\tau$ on any interval are represented by the values at the end of that interval, as indicated by their subscripts.

Expanding $H(1,\tau)$ gives

$$
\begin{aligned}
&AH(1,0)\int_0^{\tau_1} \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\mathrm{d}\tau \\
&+A\left(\frac{H(1,\tau_1)-H(1,0)}{\tau_1}\right)\int_0^{\tau_1} \tau \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\mathrm{d}\tau \\
&+\left[\frac{L^2}{\mu KD}\left(b+R_1\right)-\left.\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right|_1\right]\int_0^{\tau_1} \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\mathrm{d}\tau \\
&+A\sum_{i=1}^{j}\left[H(1,\tau_i)-\frac{\tau_i\left(H(1,\tau_{i+1})-H(1,\tau_i)\right)}{\tau_{i+1}-\tau_i}\right]\int_{\tau_i}^{\tau_{i+1}} \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\mathrm{d}\tau \\
&+A\sum_{i=1}^{j}\frac{H(1,\tau_{i+1})-H(1,\tau_i)}{\tau_{i+1}-\tau_i}\int_{\tau_i}^{\tau_{i+1}} \tau \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\mathrm{d}\tau \\
&+\sum_{i=1}^{j}\left[\frac{L^2}{\mu KD}\left(b+R_{i+1}\right)-\left.\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right|_{i+1}\right]\int_{\tau_i}^{\tau_{i+1}} \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\mathrm{d}\tau
\end{aligned}
\tag{60}
$$

Two integrals appear in the expression. They are evaluated as:

$$
\int_{\tau_i}^{\tau_{i+1}} \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\mathrm{d}\tau = \mathrm{e}^{\gamma_n t}\int_{\tau_i}^{\tau_{i+1}} \mathrm{e}^{-(\gamma_n+A)\tau}\mathrm{d}\tau = \frac{\mathrm{e}^{\gamma_n t}}{\gamma_n+A}\left[\mathrm{e}^{-(\gamma_n+A)\tau_i}-\mathrm{e}^{-(\gamma_n+A)\tau_{i+1}}\right]
\tag{61}
$$

$$
\begin{aligned}
&\int_{\tau_i}^{\tau_{i+1}} \tau \mathrm{e}^{\gamma_n(t-\tau)-A\tau}\mathrm{d}\tau = \mathrm{e}^{\gamma_n t}\int_{\tau_i}^{\tau_{i+1}} \tau \mathrm{e}^{-(\gamma_n+A)\tau}\mathrm{d}\tau \\
&=\frac{\mathrm{e}^{\gamma_n t}}{(\gamma_n+A)^2}\left\{\left[(\gamma_n+A)\tau_i+1\right]\mathrm{e}^{-(\gamma_n+A)\tau_i}-\left[(\gamma_n+A)\tau_{i+1}+1\right]\mathrm{e}^{-(\gamma_n+A)\tau_{i+1}}\right\}
\end{aligned}
\tag{62}
$$

With these expressions, Eq. (60) becomes

$$\frac{e^{\gamma_n t}}{\gamma_n + A}\left(\begin{aligned}&AH(1,0)\left[1-e^{-(\gamma_n+A)\tau_1}\right]\\&+\frac{A}{\gamma_n+A}\left(\frac{H(1,\tau_1)-H(1,0)}{\tau_1}\right)\left\{1-\left[(\gamma_n+A)\tau_1+1\right]e^{-(\gamma_n+A)\tau_1}\right\}\\&+\left[\frac{L^2}{\mu KD}(b+R_1)-\frac{H(1,\tau_1)-H(1,0)}{\tau_1}\right]\left[1-e^{-(\gamma_n+A)\tau_1}\right]\\&+A\sum_{i=1}^{j}\left[H(1,\tau_i)-\frac{\tau_i\left(H(1,\tau_{i+1})-H(1,\tau_i)\right)}{\tau_{i+1}-\tau_i}\right]\left[e^{-(\gamma_n+A)\tau_i}-e^{-(\gamma_n+A)\tau_{i+1}}\right]\\&+\frac{A}{\gamma_n+A}\sum_{i=1}^{j}\left(\frac{H(1,\tau_{i+1})-H(1,\tau_i)}{\tau_{i+1}-\tau_i}\right)\left\{\left[(\gamma_n+A)\tau_i+1\right]e^{-(\gamma_n+A)\tau_i}-\left[(\gamma_n+A)\tau_{i+1}+1\right]e^{-(\gamma_n+A)\tau_{i+1}}\right\}\\&+\sum_{i=1}^{j}\left[\frac{L^2}{\mu KD}(b+R_{i+1})-\frac{H(1,\tau_{i+1})-H(1,\tau_i)}{\tau_{i+1}-\tau_i}\right]\left[e^{-(\gamma_n+A)\tau_i}-e^{-(\gamma_n+A)\tau_{i+1}}\right]\end{aligned}\right) \quad (63)$$

where the staircase nature of dH(1,t)/dt has been incorporated in the third and sixth terms. Expression (63) provides the explicit evaluation of the integral of Eq. (53) for piecewise linear $H(1,t)$ and stepwise changing $R(t)$. The terms involving exponential functions with $-\gamma_n$ in the exponent become excessively large, even for moderate n. The equation is therefore rewritten as:

$$\frac{1}{\gamma_n + A}\left(\begin{aligned}&AH(1,0)\left[e^{\gamma_n t}-e^{\gamma_n(t-\tau_1)-A\tau_1}\right]\\&+\frac{A}{\gamma_n+A}\left(\frac{H(1,\tau_1)-H(1,0)}{\tau_1}\right)\left\{e^{\gamma_n t}-\left[(\gamma_n+A)\tau_1+1\right]e^{\gamma_n(t-\tau_1)-A\tau_1}\right\}\\&+\left[\frac{L^2}{\mu KD}(b+R_1)-\frac{H(1,\tau_1)-H(1,0)}{\tau_1}\right]\left[e^{\gamma_n t}-e^{\gamma_n(t-\tau_1)-A\tau_1}\right]\\&+A\sum_{i=1}^{j}\left[H(1,\tau_i)-\frac{\tau_i\left(H(1,\tau_{i+1})-H(1,\tau_i)\right)}{\tau_{i+1}-\tau_i}\right]\left[e^{\gamma_n(t-\tau_i)-A\tau_i}-e^{\gamma_n(t-\tau_{i+1})-A\tau_{i+1}}\right]\\&+\frac{A}{\gamma_n+A}\sum_{i=1}^{j}\left(\frac{H(1,\tau_{i+1})-H(1,\tau_i)}{\tau_{i+1}-\tau_i}\right)\left\{\left[(\gamma_n+A)\tau_i+1\right]e^{\gamma_n(t-\tau_i)-A\tau_i}-\left[(\gamma_n+A)\tau_{i+1}+1\right]e^{\gamma_n(t-\tau_{i+1})-A\tau_{i+1}}\right\}\\&+\sum_{i=1}^{j}\left[\frac{L^2}{\mu KD}(b+R_{i+1})-\frac{H(1,\tau_{i+1})-H(1,\tau_i)}{\tau_{i+1}-\tau_i}\right]\left[e^{\gamma_n(t-\tau_i)-A\tau_i}-e^{\gamma_n(t-\tau_{i+1})-A\tau_{i+1}}\right]\end{aligned}\right) \quad (64)$$

For application in hydrological models that have discrete time steps, the calculations will restart every time step. The time is reset to zero at the start of the time step, and the dimensionless duration of the time step is denoted t_1. Thus, $H(x,0)$ will be the result of the computations of the previous time step. The surface water level will have to be updated by taking the flux between the groundwater and the surface water (calculated by the groundwater model), and the river flux. This will then give $H(1,0)$. In all likelihood, $H(1, t_1)$ will be set equal to $H(1,0)$ (to be updated at the end of the time step to provide a new value for the next time step. Nevertheless, the single-time step equations will be cast such that the surface water levels at the beginning and the end of the time step can vary. The value of R will be constant during the time step.

For hydrological models, Equations (47a) through (49a) are of interest. The single time-step simplifies the integral over time in these equations, i.e. Eq. (53) and its fully evaluated form in Eq. (64). The single–time–step version of Eq. (64) reads:

$$\frac{1}{\gamma_n + A}\left(\begin{array}{l} AH(1,0)\left(\mathrm{e}^{\gamma_n t_1} - \mathrm{e}^{-At_1}\right) \\ + \dfrac{A}{\gamma_n + A}\left(\dfrac{H(1,t_1) - H(1,0)}{t_1}\right)\left\{\mathrm{e}^{\gamma_n t_1} - \left[(\gamma_n + A)t_1 + 1\right]\mathrm{e}^{-At_1}\right\} \\ + \left[\dfrac{L^2}{\mu KD}(b+R) - \dfrac{H(1,t_1) - H(1,0)}{t_1}\right]\left(\mathrm{e}^{\gamma_n t_1} - \mathrm{e}^{-At_1}\right) \end{array}\right) \tag{65}$$

In addition to the flux per unit bank length at the end of the time step it is useful to have the total amount of water (per unit bank length) that flowed from the aquifer to the surface water or vice versa during the time step:

$$\frac{Q_{tot}}{\Delta x_2}(1) = \int_0^{t_1} \frac{Q}{\Delta x_2}(1,t)\mathrm{d}t = \frac{2KD}{L}\sum_{n=0}^{\infty}\frac{\left(n+\frac{1}{2}\right)}{(-1)^n}\int_0^{t_1} M_n(t)\mathrm{d}t \tag{66}$$

With Eq. (50) the integral in this expression becomes

$$\int_0^{t_1} M_n(t)\mathrm{d}t = \pi\int_0^{t_1} \mathrm{e}^{\left\{\frac{aL^2}{KD}-\left[\left(n+\frac{1}{2}\right)\pi\right]^2\right\}\frac{t}{\mu}}\mathrm{d}t\int_0^1 \left(H(x,0)-H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x$$

$$+\frac{(-1)^n}{n+\frac{1}{2}}\int_0^{t_1}\mathrm{e}^{\frac{aL^2}{\mu KD}t}\int_0^t \mathrm{e}^{-\frac{1}{\mu}\left[\left(n+\frac{1}{2}\right)\pi\right]^2(t-\tau)-\frac{aL^2}{\mu KD}\tau}\left(\frac{aL^2}{\mu KD}H(1,\tau)+\frac{L^2}{KD}\frac{b+R(\tau)}{\mu}-\frac{\mathrm{d}H(1,\tau)}{\mathrm{d}\tau}\right)\mathrm{d}\tau\mathrm{d}t \quad (67)$$

With Eq. (64) we obtain

$$\int_0^{t_1} M_n(t)\mathrm{d}t = \frac{\pi\mu}{\frac{aL^2}{KD}-\left[\left(n+\frac{1}{2}\right)\pi\right]^2}\left(\mathrm{e}^{\left\{\frac{aL^2}{KD}-\left[\left(n+\frac{1}{2}\right)\pi\right]^2\right\}\frac{t_1}{\mu}}-1\right)\int_0^1\left(H(x,0)-H(1,0)\right)\cos\left[\left(n+\frac{1}{2}\right)\pi x\right]\mathrm{d}x$$

$$+\frac{(-1)^n}{n+\frac{1}{2}}\frac{1}{\gamma_n+A}\int_0^{t_1}\left\{\begin{array}{l} AH(1,0)\left[\mathrm{e}^{(\gamma_n+A)t}-1\right] \\ +\frac{A}{\gamma_n+A}\left(\frac{H(1,t_1)-H(1,0)}{t_1}\right)\left[\mathrm{e}^{(\gamma_n+A)t}-(\gamma_n+A)t-1\right] \\ +\left[\frac{L^2}{\mu KD}(b+R)-\frac{H(1,t_1)-H(1,0)}{t_1}\right]\left[\mathrm{e}^{(\gamma_n+A)t}-1\right]\end{array}\right\}\mathrm{d}t$$

(68)

where the evaluation of the integral over x is given by Eq. (57). For the integral over time we find:

$$\int_0^{t_1}\left\{\begin{array}{l} AH(1,0)\left[\mathrm{e}^{(\gamma_n+A)t}-1\right] \\ +\frac{A}{\gamma_n+A}\frac{H(1,t_1)-H(1,0)}{t_1}\left[\mathrm{e}^{(\gamma_n+A)t}-(\gamma_n+A)t-1\right] \\ +\left[\frac{L^2}{\mu KD}(b+R)-\frac{H(1,t_1)-H(1,0)}{t_1}\right]\left[\mathrm{e}^{(\gamma_n+A)t}-1\right]\end{array}\right\}\mathrm{d}t$$

$$=\frac{AH(1,0)}{\gamma_n+A}\left[\mathrm{e}^{(\gamma_n+A)t_1}-1\right]-AH(1,0)t_1$$

$$+\frac{A}{(\gamma_n+A)^2}\frac{H(1,t_1)-H(1,0)}{t_1}\left[\mathrm{e}^{(\gamma_n+A)t}-1\right]-\frac{A}{2}\frac{H(1,t_1)-H(1,0)}{t_1}t_1-\frac{A\left(H(1,t_1)-H(1,0)\right)}{\gamma_n+A}$$

$$+\left[\frac{\frac{L^2}{\mu KD}(b+R)-\frac{H(1,t_1)-H(1,0)}{t_1}}{\gamma_n+A}\right]\left[\mathrm{e}^{(\gamma_n+A)t_1}-1\right]-\left[\frac{L^2}{\mu KD}(b+R)-\frac{H(1,t_1)-H(1,0)}{t_1}\right]t_1$$

(69)

After some rearranging, this simplifies to

$$\int_0^{t_1} \left\{ \begin{array}{l} AH(1,0)\left[\mathrm{e}^{(\gamma_n + A)t} - 1\right] \\ + \dfrac{A}{\gamma_n + A} \dfrac{H(1,t_1) - H(1,0)}{t_1} \left[\mathrm{e}^{(\gamma_n + A)t} - (\gamma_n + A)t - 1\right] \\ + \left[\dfrac{L^2}{\mu KD}(b + R) - \dfrac{H(1,t_1) - H(1,0)}{t_1}\right]\left[\mathrm{e}^{(\gamma_n + A)t} - 1\right] \end{array} \right\} \mathrm{d}t$$

$$= \left[AH(1,0) + \left(\frac{A}{\gamma_n + A} - 1\right)\frac{H(1,t_1) - H(1,0)}{t_1} + \frac{L^2(b + R)}{\mu KD}\right] \frac{\mathrm{e}^{(\gamma_n + A)t_1} - 1}{\gamma_n + A} \tag{70}$$

$$- \left[AH(1,0) + \frac{L^2(b + R)}{\mu KD}\right] t_1 + \left(1 - \frac{A}{2} - \frac{A}{\gamma_n + A}\right)\left(H(1,t_1) - H(1,0)\right)$$

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Appendix. Linearizing the dimensional equation and converting it to dimensionless form

N.B. The material covered here is taken verbatim from de Rooij, 2012.

Invoking the Dupuit assumptions for groundwater flow eliminates vertical gradients in the hydraulic head, and only the horizontal coordinates remain. Phreatic aquifers may receive recharge from the unsaturated zone above that is independent of the local hydraulic head and may exchange water with a deeper aquifer if the separating aquitard is somewhat permeable. Such exchange fluxes are assumed here to be proportional to the local hydraulic head. For a uniform, isotropic, phreatic aquifer overlying a level aquitard, the governing PDE then becomes:

$$\mu \frac{\partial H}{\partial t^*} = K\left[\frac{\partial}{\partial x_1}\left(H\frac{\partial H}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(H\frac{\partial H}{\partial x_2}\right)\right] + aH + b + R \tag{A1}$$

where x_1 and x_2 [L] are the horizontal coordinates, t^* is time [T], H is the hydraulic head [L], defined with respect to the top of the underlying aquitard, K [LT^{-1}] is the hydraulic conductivity, R [LT^{-1}] is the recharge or loss to evapotranspiration (R may be time dependent), μ is the storage coefficient (occasionally termed drainable porosity for a phreatic aquifer, e.g., van Schilfgaarde, 1974), and a [T^{-1}] (≤ 0) and b [LT^{-1}] are constants determining the exchange with the deeper aquifer. If the deeper aquifer has a constant and uniform hydraulic head H_2, $-a^{-1}$ is the resistance of the aquitard, and $b = -aH_2$. Equation (1) is the Boussinesq equation with additional production terms and does not have a general analytical solution. To make the equation analytically tractable it needs to be linearized by assuming that the variation in H is small with respect to H, and that μ is a constant:

$$\frac{\partial H}{\partial t^*} = \frac{KD}{\mu}\left(\frac{\partial^2 H}{\partial {x_1}^2} + \frac{\partial^2 H}{\partial {x_2}^2}\right) + \frac{a}{\mu}H + \frac{b+R}{\mu} \tag{A2}$$

where D [L] is the constant water level above the aquitard. The combination of the Dupuit assumptions and the linearization has a sound footing in classical drainage theory (e.g. Dumm, 1954; Kraijenhoff van de Leur, 1958; van Schilfgaarde, 1970; Wesseling, 1979). Van Schilfgaarde (1974) gives a thoughtful discussion of the assumptions underlying the above linearization.

To analyze flow towards parallel drains, ditches, or streams with spacing 2*L* [L], we drop the second horizontal coordinate since the flow lines are all perpendicular to it. We also make the independent variables dimensionless by the following transformations:

$$x = \frac{x_1}{L} \qquad \text{(A3)}$$

$$t = \frac{KD}{L^2} t^* \qquad \text{(A4)}$$

to obtain:

$$\frac{\partial H}{\partial t} = \frac{1}{\mu}\frac{\partial^2 H}{\partial x^2} + \frac{L^2}{KD}\frac{a}{\mu}H + \frac{L^2}{KD}\frac{b+R}{\mu} \qquad \text{(A5)}$$