Hydroinformatik II:
Partielle Differentialgleichungen (PDEs)

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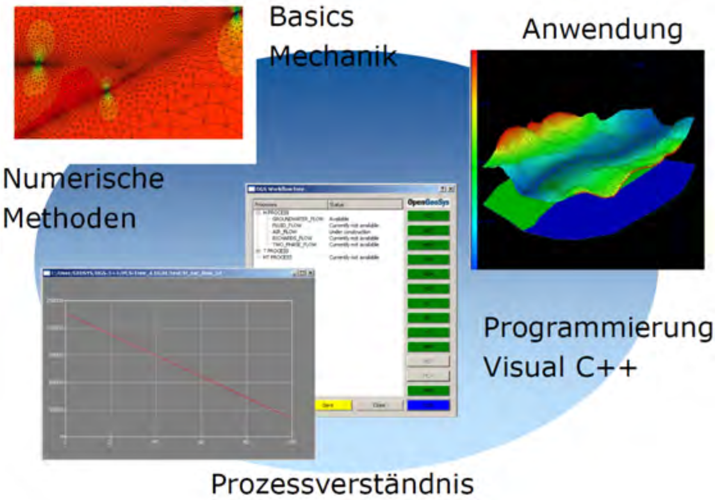
Dresden, 05. Mai 2017

Vorlesungsplan Hydroinformatik II SoSe 2017

#	Datum	Thema
01	07.04.2017	Einführung
02	07.04.2017	Grundlagen: Kontinuumsmechanik
–	14.04.2017	Ostern
03	21.04.2017	Grundlagen: Hydromechanik
04	28.04.2017	Qt Installation (2017) Robert Schlick
05	05.05.2017	Grundlagen: Hydromechanik / Partielle Differentialgleichung
06	12.05.2017	Qt Einführung
07	19.05.2017	Qt Übung: Funktionsrechner; Grundlagen Numerik
08	26.05.2017	Numerik: (exp) Finite Differenzen Methode
09	02.06.2017	Numerik: (imp) Finite Differenzen Methode
–	09.06.2017	Pfingsten
10	16.06.2017	Gerinnehydraulik: Theorie - Grundlagen
11	23.06.2017	HW: Gerinnehydraulik: Programmierung, Übung 1
12	30.06.2017	Gerinnehydraulik: Programmierung, Übung 2
13	07.07.2017	Einführung - Grundwassermodellierung (MW)
14	14.07.2017	Kurs-Zusammenfassung, Ausblick und Beleg

Konzept

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Inhalte

- Link zur letzten Vorlesung: Dimensionsanalyse

- Konzept
- Partielle Differentialgleichungen
- Klassifikation
- Beispiele
- Anfangs- und Randbedingungen

- TEXfür den Beleg

Dimensionsanalyse

Navier-Stokes-Gleichung

Mathematical Classification (1.5)

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, ..., \frac{\partial^n \psi}{\partial x_i^n}) = 0 \quad , \quad i = 3 \tag{1}$$

where L is s differential operator. Second-order PDE with two independent variables are given by

$$A\frac{\partial^2 \psi}{\partial x^2} + B\frac{\partial^2 \psi}{\partial x \partial y} + C\frac{\partial^2 \psi}{\partial y^2} + D\frac{\partial \psi}{\partial x} + E\frac{\partial \psi}{\partial y} + F\psi + G = 0 \tag{2}$$

Second-order PDEs with more independent variables can be classified by examination of the eigenvalues of the matrix a_{ij}.

$$\sum_i \sum_j a_{ij} \frac{\partial \psi^2}{\partial x_i \partial x_j} + G = 0 \tag{3}$$

Mathematical Classification (1.5)

PDE type	Discriminant	Eigenvalues	Canonical form	Example
Elliptic	$B^2 - 4AC < 0$ complex characteristics	$\forall\lambda > 0$ equal signs	$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0$	Laplace equation
Parabolic	$B^2 - 4AC = 0$	$\exists\lambda = 0$	$\frac{\partial^2\psi}{\partial\eta^2} = G$	Diffusion, Burgers equations
Hyperbolic	$B^2 - 4AC > 0$ real characteristics	$\exists\lambda < 0$ different signs	$\frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta^2} = 0$	Wave equation

General Balance Equation (1.1.7)

- Integral form

$$\int_\Omega \frac{d\psi}{dt} d\Omega = \int_\Omega \frac{\partial\psi}{\partial t} d\Omega + \int_\Omega \nabla \cdot (\mathbf{v}\psi) d\Omega - \int_\Omega \nabla \cdot (\mathbf{D}^\psi \nabla\psi) d\Omega = \int_\Omega Q^\psi d\Omega \quad (4)$$

- Differential form

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla\psi) = \mathbf{Q}^\psi \quad (5)$$

PDE: Definition

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = \mathbf{Q}^{\psi} \tag{6}$$

A common formulation of a PDE in $\mathcal{R}^3$ is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, ..., \frac{\partial^n \psi}{\partial x_i^n}) = 0 \quad , \quad i = 3 \tag{7}$$

where L is s differential operator.

PDE: Klassifikation

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi}\nabla\psi) = \mathbf{Q}^{\psi} \quad (8)$$

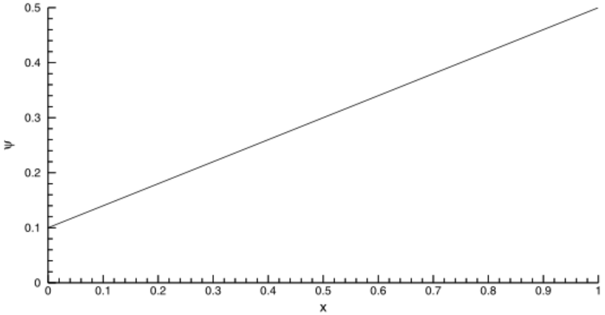
Physical problem	Math. problem	Examples
Equilibrium problems	Elliptic equations	Irrotational incompressible flow Inviscid incompressible flow Steady state heat conduction
Propagation problems (infinite propagation speed)	Parabolic equations	Unsteady viscous flow Transient heat transfer
Propagation problems (finite propagation speed)	Hyperbolic equations	Wave propagation (vibration) Inviscid supersonic flow

- Parabolisch: Diffusion, Gerinne (nichtlinear)
- Elliptisch: Grundwasser (stationär)

PDE: Elliptic Equation 1-D

$$\frac{d^2\psi}{dx^2} = 0 \quad (9)$$

$$\psi = ax + b \quad (10)$$



PDE: Elliptic Equation 2-D

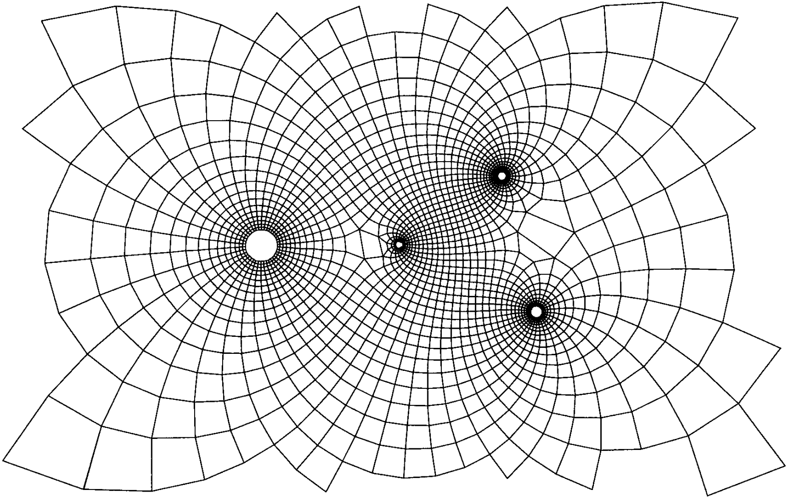
The prototype of an elliptic equation is the Laplace equation.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{11}$$

By substitution it can be easily verified that the exact solution of the Laplace equation is

$$\psi = sin(\pi x)exp(-\pi y) \tag{12}$$

PDE: Elliptic Equation 2-D



PDE: Parabolic Equation 1-D

$$\frac{\partial \psi}{\partial t} = \alpha \frac{\partial^2 \psi}{\partial x^2} \tag{13}$$

Multiple solutions:

$$\psi(t, x) = sin(\sqrt{\pi\alpha}x)exp(-\pi t) \quad ??? \tag{14}$$

$$\psi(t, x) = sin(\frac{\pi}{\sqrt{\alpha}}x)exp(-\pi^2 t) \tag{15}$$

$$\psi(t, x) = sin(\pi x)exp(-\alpha\pi^2 t) \tag{16}$$

PDE: Parabolic Equation 1-D

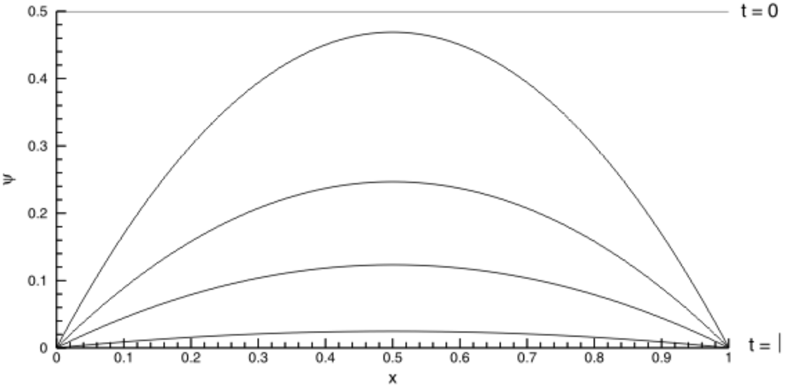


Abbildung: Solution of a parabolic equation

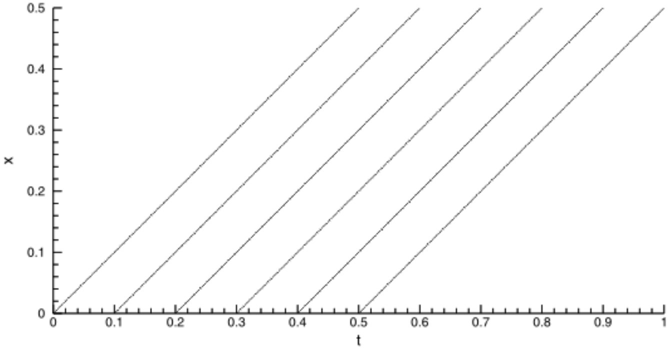
PDE: Hyperbolic Equation 1-D

$$\frac{\partial^2 \psi}{\partial^2 t} - c^2 \frac{\partial^2 \psi}{\partial^2 x} = 0 \tag{17}$$

$$\psi(t, x) = a \cos(\frac{\pi c t}{L}) \sin(\frac{\pi x}{L}) \tag{18}$$

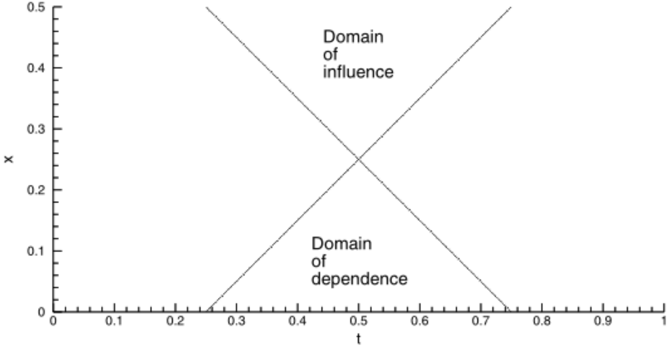
PDE: Hyperbolic Equation 1-D - Characteristics

$$A\frac{\partial\psi}{\partial t} - B\frac{\partial\psi}{\partial x} = 0 \tag{19}$$



PDE: Hyperbolic Equation 1-D - Characteristics

$$A\frac{\partial\psi}{\partial t} - B\frac{\partial\psi}{\partial x} = 0 \qquad (20)$$



PDE: Equation Types

The following table gives typical examples of balance equations for the denoted quantities and their PDE types.

Physical meaning	Equation structure	Examples
Continuity	$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$	Laplace equation
Mass/energy	$\frac{\partial\psi}{\partial t} + u\frac{\partial\psi}{\partial x} - \alpha\frac{\partial^2\psi}{\partial x^2} = 0$	Fokker-Planck equation
Momentum	$\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}[\alpha(\psi)\frac{\partial\psi}{\partial x}] = 0$	Navier-Stokes equation

Boundary Conditions I

The following table gives an overview on common boundary condition types and its mathematical representation.

Tabelle: Boundary conditions types

Type of BC	Mathematical Meaning	Physical Meaning
Dirichlet type	ψ	prescribed value potential surface
Neumann type	$\nabla\psi$	prescribed flux stream surface
Cauchy type	$\psi + A\nabla\psi$	resistance between potential and stream surface

Boundary Conditions II

To describe conditions at boundaries we can use flux expressions of conservation quantities.

Tabelle: Fluxes through surface boundaries

Quantity	Flux term
Mass	$\rho \mathbf{v}$
Momentum	$\rho \mathbf{v}\mathbf{v} - \sigma$
Energy	$\rho e \mathbf{v} - \lambda \nabla \mathbf{T}$

TEX1#3

```
%==============================================================
\documentclass[twoside]{report} % double side
%%\documentclass[]{report} % single side
%--------------------------------------------------------------
% Pakete
\usepackage[dvips]{graphicx}
\usepackage{epsfig}
\usepackage{german}        % Verwenden der deutschen Trennmuster
\usepackage[ansi]{umlaute} % Unterstuetzen von deutschen Umlauten
\hyphenation{me-cha-nik}   % Trennmuster fuer Ausnahmefaelle
%--------------------------------------------------------------
% Formatierung
\setlength{\parindent}{0pt} % Absaetze nicht einruecken
\setlength{\parskip}{5pt plus 2pt minus 1pt}
\setcounter{secnumdepth}{5} %
\usepackage{fheading}       % Seitenkopf gestalten
\pagestyle{fancy}
\lhead[\fancyplain{}{\footnotesize\textsf\thepage}]%
      {\fancyplain{}{\footnotesize\textsf\rightmark}}
\rhead[\fancyplain{}{\footnotesize\textsf\leftmark}]%
      {\fancyplain{}{\footnotesize\textsf\thepage}}
\cfoot{}
...
```

TEX2#3

```
%-----------------------------------------------------------
% Makros
\def \UVec {\mathbf x}
\def \AMat {\mathbf A}
\def \RHS {\mathbf b}
\def \Jacobian {\mathbf J}
\def \E {$\Rightarrow$}
%
\newcommand{\red}[1]{{\color{red}#1}}
%
\include{makros/rfsdef}
\include{makros/rfddef}
\include{makros/stnddef}
% --- Fonts
\include{fonts}
```

TEX3#3

```
%===========================================================
\begin{document}
%-----------------------------------------------------------
\thispagestyle{empty}
\include{titel}
%-----------------------------------------------------------
\include{exam2012-13WS_Beleg}
%-----------------------------------------------------------
% Citations
\nocite{Kol:2002}
\bibliographystyle{plain} % unsrt}
\bibliography{software}
%-----------------------------------------------------------
% Inhaltsverzeichnis
\tableofcontents
%-----------------------------------------------------------
\end{document}
%===========================================================
```