Hydroinformatik II:

Grundlagen der Kontinuumsmechanik

V3

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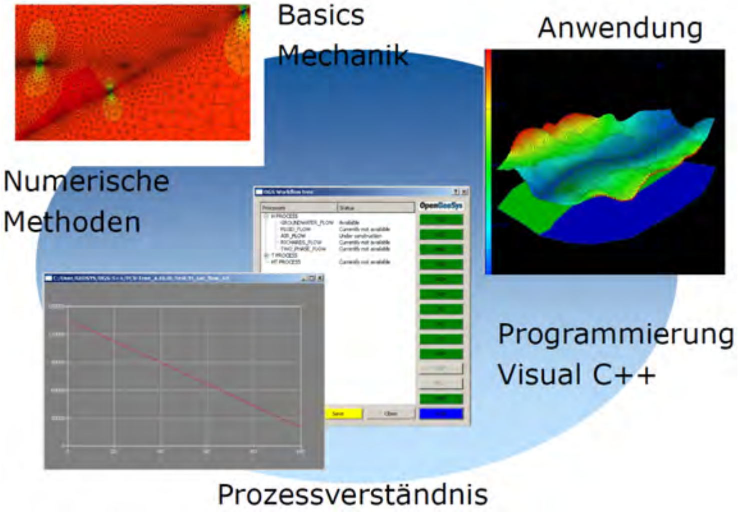
Vorlesungsplan Hydroinformatik II SoSe 2017

#	Datum	Thema
01	07.04.2017	Einführung
02	07.04.2017	Grundlagen: Kontinuumsmechanik
–	14.04.2017	Ostern
03	21.04.2017	Grundlagen: Hydromechanik
04	28.04.2017	Qt Installation (2017) Robert Schlick
05	05.05.2017	Grundlagen: Hydromechanik / Partielle Differentialgleichung
06	12.05.2017	Qt Einführung
07	19.05.2017	Qt Übung: Funktionsrechner; Grundlagen Numerik
08	26.05.2017	Numerik: (exp) Finite Differenzen Methode
09	02.06.2017	Numerik: (imp) Finite Differenzen Methode
–	09.06.2017	Pfingsten
10	16.06.2017	Gerinnehydraulik: Theorie - Grundlagen
11	23.06.2017	HW: Gerinnehydraulik: Programmierung, Übung 1
12	30.06.2017	Gerinnehydraulik: Programmierung, Übung 2
13	07.07.2017	Einführung - Grundwassermodellierung (MW)
14	14.07.2017	Kurs-Zusammenfassung, Ausblick und Beleg

Konzept

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$

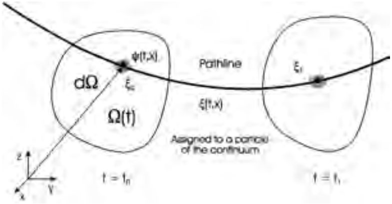


Inhalte

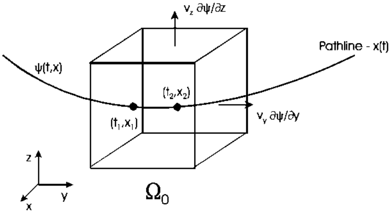
- Erhaltungsgrößen
- Massenerhaltung
- Fluidmassenerhaltung
- Diffusion
- Impulserhaltung
- Spannungen
- Fluiddruck
- Strömungsprobleme

General Balance Equation

Lagrange



Euler



General Balance Equation

$$\frac{d}{dt}\int_\Omega \psi \, d\Omega = \frac{\partial}{\partial t}\int_\Omega \psi \, d\Omega + \oint_{\partial\Omega} \mathbf{\Phi}^\psi \cdot d\mathbf{S} \tag{1}$$
$$= \frac{\partial}{\partial t}\int_\Omega \psi \, d\Omega + \int_\Omega \nabla \cdot \mathbf{\Phi}^\psi d\Omega$$

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot \mathbf{\Phi}^\psi \tag{2}$$
$$= \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla\psi)$$
$$= Q^\psi$$

Notation

Conservation Quantities (1.1.2)

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \tag{3}$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ, momentum density $\rho\mathbf{v}$ or energy density e.

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
Linear momentum	$\mathbf{m}$	Linear momentum density	$\rho\mathbf{v}$
Energy	E	Energy density	$e = \rho i + \frac{1}{2}\rho v^2$

(Phase) Mass Conservation

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla \psi) = Q^{\psi} \qquad (4)$$

The differential equation of mass conservation in divergence form becomes

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \qquad (5)$$

Partial differentiation of the above equation gives

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0 \qquad (6)$$

(Phase) Mass Conservation

Using the material (or convective) derivative the mass conservation equation can be rewritten as

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{v} \tag{7}$$

Note, above convective form of mass conservation equation becomes zero only for incompressible flows, i.e.

$$\frac{\partial \rho}{\partial t} = 0 \tag{8}$$

requires divergence-free flow.

$$\nabla \cdot \mathbf{v} = 0 \tag{9}$$

From eqn. (6) results that the above expression is the continuity equation for a homogeneous fluid ($\rho = \text{const}$).

Conservation Quantities (1.1.2)

The amount of a quantity in a defined volume Ω is given by

$$\Psi = \int_{\Omega} \psi d\Omega(t) \tag{10}$$

where Ψ is an extensive conservation quantity (i.e. mass, momentum, energy) and ψ is the corresponding intensive conservation quantity such as mass density ρ, momentum density $\rho\mathbf{v}$ or energy density e.

Extensive quantity	Symbol	Intensive quantity	Symbol
Mass	M	Mass density	ρ
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Momentum Conservation

$$\psi = \rho \mathbf{v}$$

$$\frac{\partial}{\partial t} \int\limits_{\Omega} \rho \mathbf{v} d\Omega + \oint\limits_{\partial\Omega} \mathbf{\Phi}^m \cdot d\mathbf{S} = \int\limits_{\Omega} \rho \mathbf{f} d\Omega \tag{11}$$

Flux term: The advective momentum flux is defined as

$$\mathbf{\Phi}^m = (\rho \mathbf{v}) \otimes \mathbf{v} = (\rho \mathbf{v}) \mathbf{v} \tag{12}$$

$$\mathbf{F} = \int\limits_{\Omega} \rho \mathbf{f} d\Omega = \int\limits_{\Omega} \rho (\mathbf{f}^e + \mathbf{f}^i) d\Omega = \underbrace{\int\limits_{\Omega} \rho \mathbf{f}^e d\Omega}_{\text{External forces}} + \underbrace{\oint\limits_{\partial\Omega} \boldsymbol{\sigma} : d\mathbf{S}}_{\text{Internal forces}} \tag{13}$$

Momentum Conservation

Substituting now flux and source terms of momentum we obtain

$$\frac{\partial}{\partial t}\int\limits_{\Omega}\rho\mathbf{v}\,d\Omega + \oint\limits_{\partial\Omega}\rho\mathbf{v}(\mathbf{v}\cdot d\mathbf{S}) = \int\limits_{\Omega}\rho\mathbf{f}^{e}\,d\Omega + \oint\limits_{\partial\Omega}\boldsymbol{\sigma}\,d\mathbf{S} \tag{14}$$

Applying the Gauss-Ostrogradskian theorem to the surface integrals

$$\begin{aligned}\oint\limits_{\partial\Omega}\rho\mathbf{v}(\mathbf{v}\cdot d\mathbf{S}) &= \int\limits_{\Omega}\nabla\cdot(\rho\mathbf{v}\mathbf{v})\,d\Omega \\ \oint\limits_{\partial\Omega}\boldsymbol{\sigma}\,d\mathbf{S} &= \int\limits_{\Omega}\nabla\cdot\boldsymbol{\sigma}\,d\Omega \end{aligned} \tag{15}$$

Momentum Conservation

The differential form of the momentum conservation law is then

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \rho \mathbf{f}^e + \nabla \cdot \boldsymbol{\sigma} \tag{16}$$

The above equation is now extended by partial integration

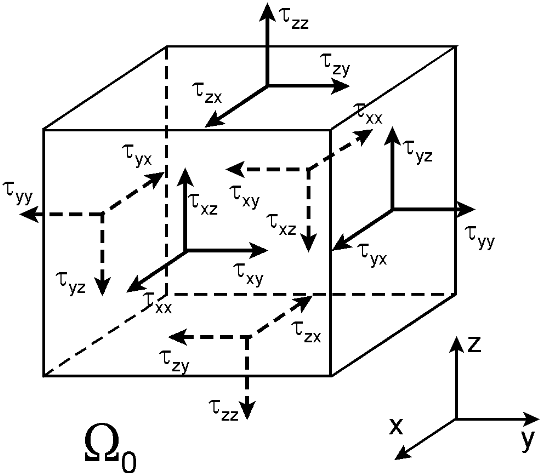
$$\begin{aligned} \rho\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t} + (\rho \mathbf{v}) \cdot \nabla \mathbf{v} + \mathbf{v}\nabla \cdot (\rho \mathbf{v}) \\ = \rho[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}] + \mathbf{v}[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})] \\ = \rho \mathbf{f}^e + \nabla \cdot \boldsymbol{\sigma} \end{aligned} \tag{17}$$

Using the mass conservation equation (5) and dividing by ρ we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} \tag{18}$$

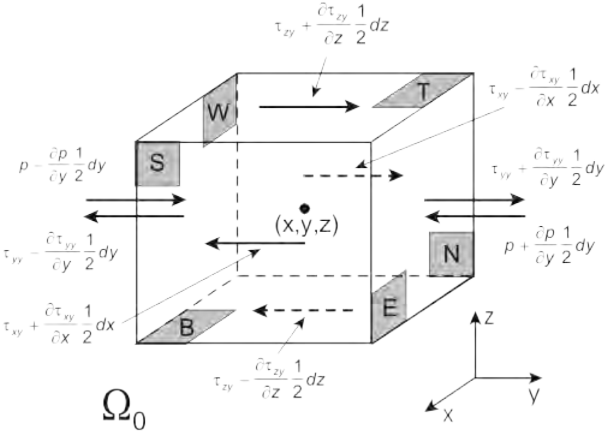
Momentum Conservation: Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \quad , \quad \boldsymbol{\tau} = \nu \nabla \mathbf{v} \tag{19}$$



Momentum Conservation: Stress Tensor

$$\boldsymbol{\tau} = \nu \nabla \mathbf{v} \tag{20}$$



Fluid Momentum Balance

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e + \frac{1}{\rho}\nabla \cdot \boldsymbol{\sigma} \tag{21}$$

In index notation the above vector equation is written as

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \frac{1}{\rho}(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}) \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= \frac{1}{\rho}(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}) \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= g + \frac{1}{\rho}(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z})
\end{aligned} \tag{22}$$

with $u = v_x, v = v_y, w = v_z$ and $\mathbf{f}^e = \mathbf{g}$.

Flow Equations - Systematic

Stress Tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + \tau \tag{23}$$

Navier-Stokes Equation

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v}} \tag{24}$$

Euler Equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}^e - \frac{1}{\rho}\nabla p \tag{25}$$

Stokes Equation

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} \tag{26}$$

Darcy Equations

$$0 = \mathbf{f}^e - \frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} \tag{27}$$

(Component) Mass Conservation

$$\psi = \rho_k = C_k \tag{28}$$

$$\frac{dC_k}{dt} = \frac{\partial C_k}{\partial t} + \nabla \cdot (\mathbf{v} C_k) - \nabla \cdot (\mathbf{D_k} \nabla C_k) = Q_k \tag{29}$$