

Hydroinformatik II: Partielle Differentialgleichungen (PDEs)

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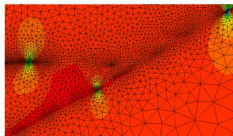
Dresden, 29. April 2015

Vorlesungsplan Hydroinformatik II SoSe 2016

#	Datum	Thema
01	08.04.2016	Einführung
02	15.04.2016	Grundlagen: Kontinuumsmechanik
03	22.04.2016	Grundlagen: Hydromechanik
04	29.04.2016	Grundlagen: Partielle Differentialgleichungen / T _E X
06	06.05.2016	HW: Qt Installation (2016)
07	13.05.2016	Qt Übung: Funktionsrechner
08	20.05.2016	Pfingsten
09	27.05.2016	Numerik: (exp) Finite Differenzen Methode
10	03.06.2016	HW / Vertretung: (DR USA)
11	10.06.2016	Numerik: (imp) Finite Differenzen Methode
12	17.06.2016	Gerinnehydraulik: Theorie - Grundlagen
13	24.06.2016	Gerinnehydraulik: Programmierung, Übung 1
14	01.07.2016	Gerinnehydraulik: Programmierung, Übung 2
15	08.07.2016	Gerinnehydraulik: Programmierung, Übung 3
16	15.07.2016	Kurs-Zusammenfassung, Ausblick und Beleg

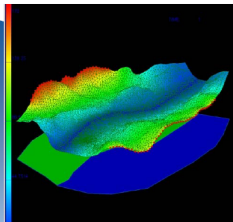
Konzept

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$

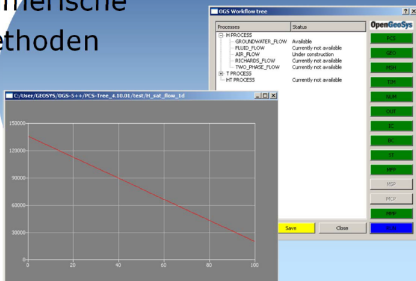


Basics
Mechanik

Anwendung



Numerische
Methoden



Programmierung
Visual C++

Prozessverständnis

Inhalte

- ▶ Link zur letzten Vorlesung: Dimensionsanalyse
- ▶ Konzept
- ▶ Partielle Differentialgleichungen
- ▶ Klassifikation
- ▶ Beispiele
- ▶ Anfangs- und Randbedingungen
- ▶ T_EX für den Beleg

Dimensionsanalyse

Navier-Stokes-Gleichung

Mathematical Classification (1.5)

A common formulation of a PDE in \mathcal{R}^3 is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial \psi}{\partial x_i}, \dots, \frac{\partial^n \psi}{\partial x_i^n}) = 0 \quad , \quad i = 3 \quad (1)$$

where L is a differential operator. Second-order PDE with two independent variables are given by

$$A \frac{\partial^2 \psi}{\partial x^2} + B \frac{\partial^2 \psi}{\partial x \partial y} + C \frac{\partial^2 \psi}{\partial y^2} + D \frac{\partial \psi}{\partial x} + E \frac{\partial \psi}{\partial y} + F \psi + G = 0 \quad (2)$$

Second-order PDEs with more independent variables can be classified by examination of the eigenvalues of the matrix a_{ij} .

$$\sum_i \sum_j a_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} + G = 0 \quad (3)$$

Mathematical Classification (1.5)

PDE type	Discriminant	Eigenvalues	Canonical form	Example
Elliptic	$B^2 - 4AC < 0$ complex characteristics	$\forall \lambda > 0$ equal signs	$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0$	Laplace equation
Parabolic	$B^2 - 4AC = 0$	$\exists \lambda = 0$	$\frac{\partial^2 \psi}{\partial \eta^2} = G$	Diffusion, Burgers equations
Hyperbolic	$B^2 - 4AC > 0$ real characteristics	$\exists \lambda < 0$ different signs	$\frac{\partial^2 \psi}{\partial \xi^2} - \frac{\partial^2 \psi}{\partial \eta^2} = 0$	Wave equation

General Balance Equation (1.1.7)

- ▶ Integral form

$$\begin{aligned}
 & \int_{\Omega} \frac{d\psi}{dt} d\Omega = \\
 & \int_{\Omega} \frac{\partial\psi}{\partial t} d\Omega + \int_{\Omega} \nabla \cdot (\mathbf{v}\psi) d\Omega - \int_{\Omega} \nabla \cdot (\mathbf{D}^{\psi} \nabla\psi) d\Omega = \\
 & \int_{\Omega} \mathbf{Q}^{\psi} d\Omega
 \end{aligned} \tag{4}$$

- ▶ Differential form

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^{\psi} \nabla\psi) = \mathbf{Q}^{\psi} \tag{5}$$

PDE: Definition

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla\psi) = \mathbf{Q}^\psi \quad (6)$$

A common formulation of a PDE in \mathcal{R}^3 is

$$L(\psi) = F(t, x_i, \psi, \frac{\partial\psi}{\partial x_i}, \dots, \frac{\partial^n\psi}{\partial x_i^n}) = 0 \quad , \quad i = 3 \quad (7)$$

where L is a differential operator.

PDE: Klassifikation

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) - \nabla \cdot (\mathbf{D}^\psi \nabla\psi) = \mathbf{Q}^\psi \quad (8)$$

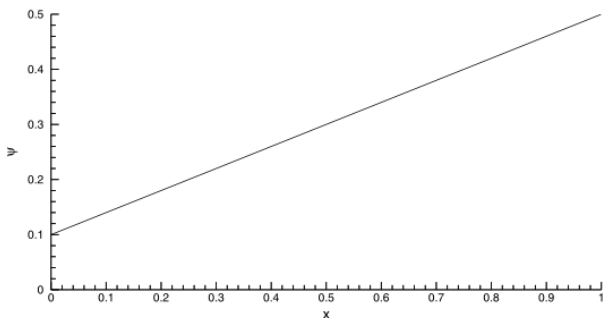
Physical problem	Math. problem	Examples
Equilibrium problems	Elliptic equations	Irrotational incompressible flow Inviscid incompressible flow Steady state heat conduction
Propagation problems (infinite propagation speed)	Parabolic equations	Unsteady viscous flow Transient heat transfer
Propagation problems (finite propagation speed)	Hyperbolic equations	Wave propagation (vibration) Inviscid supersonic flow

- ▶ Parabolisch: Diffusion, Gerinne (nichtlinear)
- ▶ Elliptisch: Grundwasser (stationär)

PDE: Elliptic Equation 1-D

$$\frac{d^2\psi}{dx^2} = 0 \quad (9)$$

$$\psi = ax + b \quad (10)$$



PDE: Elliptic Equation 2-D

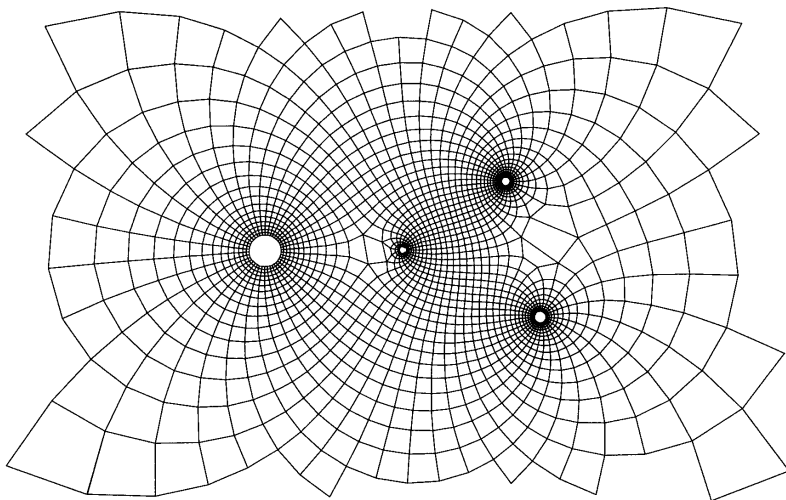
The prototype of an elliptic equation is the Laplace equation.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (11)$$

By substitution it can be easily verified that the exact solution of the Laplace equation is

$$\psi = \sin(\pi x) \exp(-\pi y) \quad (12)$$

PDE: Elliptic Equation 2-D



PDE: Parabolic Equation 1-D

$$\frac{\partial \psi}{\partial t} = \alpha \frac{\partial^2 \psi}{\partial x^2} \quad (13)$$

Multiple solutions:

$$\psi(t, x) = \sin(\sqrt{\pi \alpha x}) \exp(-\pi t) \quad (14)$$

$$\psi(t, x) = \sin(\pi x) \exp(-\pi^2 t) \quad (15)$$

PDE: Parabolic Equation 1-D

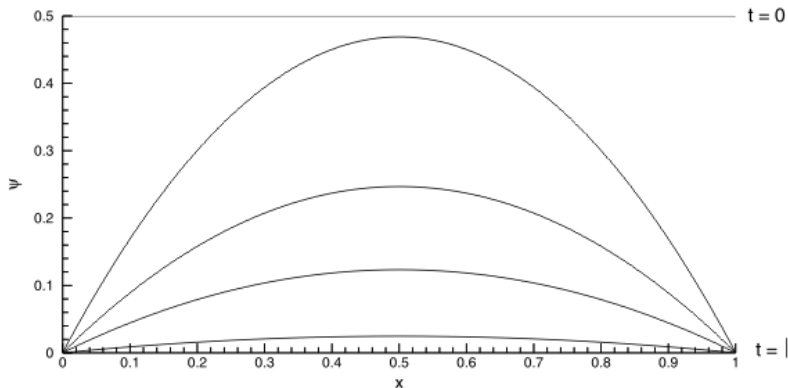


Abbildung: Solution of a parabolic equation

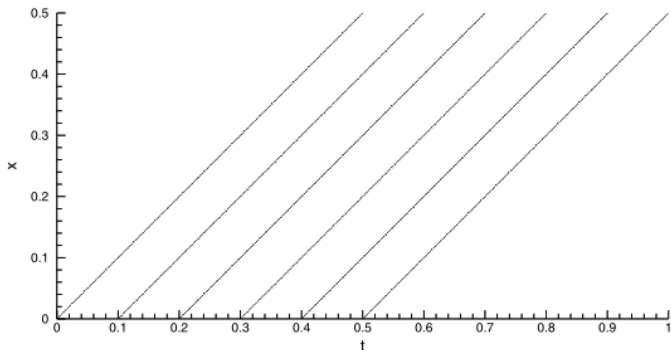
PDE: Hyperbolic Equation 1-D

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0 \quad (16)$$

$$\psi(t, x) = a \cos\left(\frac{\pi ct}{L}\right) \sin\left(\frac{\pi x}{L}\right) \quad (17)$$

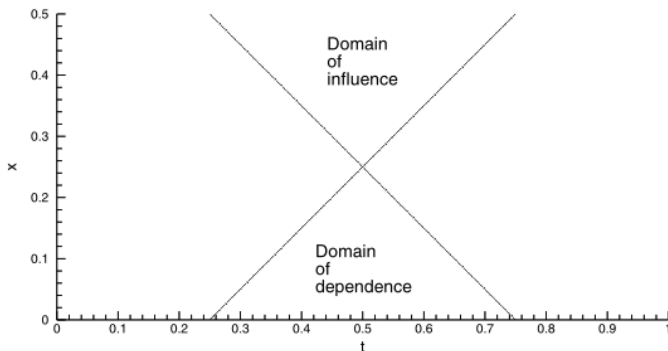
PDE: Hyperbolic Equation 1-D - Characteristics

$$A \frac{\partial \psi}{\partial t} - B \frac{\partial \psi}{\partial x} = 0 \quad (18)$$



PDE: Hyperbolic Equation 1-D - Characteristics

$$A \frac{\partial \psi}{\partial t} - B \frac{\partial \psi}{\partial x} = 0 \quad (19)$$



PDE: Equation Types

The following table gives typical examples of balance equations for the denoted quantities and their PDE types.

Physical meaning	Equation structure	Examples
Continuity	$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$	Laplace equation
Mass/energy	$\frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} - \alpha \frac{\partial^2 \psi}{\partial x^2} = 0$	Fokker-Planck equation
Momentum	$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x} \left[\alpha(\psi) \frac{\partial \psi}{\partial x} \right] = 0$	Navier-Stokes equation

Boundary Conditions I

The following table gives an overview on common boundary condition types and its mathematical representation.

Tabelle: Boundary conditions types

Type of BC	Mathematical Meaning	Physical Meaning
Dirichlet type	ψ	prescribed value potential surface
Neumann type	$\nabla\psi$	prescribed flux stream surface
Cauchy type	$\psi + A\nabla\psi$	resistance between potential and stream surface

Boundary Conditions II

To describe conditions at boundaries we can use flux expressions of conservation quantities.

Table: Fluxes through surface boundaries

Quantity	Flux term
Mass	$\rho \mathbf{v}$
Momentum	$\rho \mathbf{v} \mathbf{v} - \sigma$
Energy	$\rho e \mathbf{v} - \lambda \nabla \mathbf{T}$

T_EX1#3

```

%=====
\documentclass[twoside]{report} % double side
%%\documentclass[]{report} % single side
%-----
% Pakete
\usepackage[dvips]{graphicx}
\usepackage{epsfig}
\usepackage{german} % Verwenden der deutschen Trennmuster
\usepackage[ansi]{umlaute} % Unterstuetzen von deutschen Umlauten
\hyphenation{me-cha-nik} % Trennmuster fuer Ausnahmefaelle
%-----
% Formatierung
\setlength{\parindent}{0pt} % Absaetze nicht einruecken
\setlength{\parskip}{5pt plus 2pt minus 1pt}
\setcounter{secnumdepth}{5} %
\usepackage{fheading} % Seitenkopf gestalten
\pagestyle{fancy}
\lhead[\fancyplain{}]{\footnotesize\textsf\thepage}}%
    {\fancyplain{}{\footnotesize\textsf\rightmark}}
\rhead[\fancyplain{}]{\footnotesize\textsf\leftmark}}%
    {\fancyplain{}{\footnotesize\textsf\thepage}}
\cfoot{}
...

```

TEX2#3

```
%-----  
% Makros  
\def \UVec {\mathbf x}  
\def \AMat {\mathbf A}  
\def \RHS {\mathbf b}  
\def \Jacobian {\mathbf J}  
\def \E {$\rightarrow$}  
%  
\newcommand{\red}[1]{\color{red}#1}  
%  
\include{makros/rfsdef}  
\include{makros/rfddef}  
\include{makros/stnddef}  
% --- Fonts  
\include{fonts}
```

TEX3#3

```
%=====
\begin{document}
%-----
\thispagestyle{empty}
\include{titel}
%-----
\include{exam2012-13WS_Beleg}
%-----
% Citations
\nocite{Kol:2002}
\bibliographystyle{plain} % unsrt}
\bibliography{software}
%-----
% Inhaltsverzeichnis
\tableofcontents
%-----
\end{document}
%=====
```