

Hydroinformatik II

"Prozesssimulation und Systemanalyse"

BHYWI-08-08 © 2020

Finite-Differenzen-Methode

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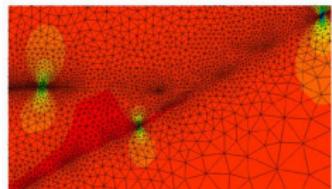
12.06.2020 - Dresden

Zeitplan: Hydroinformatik II

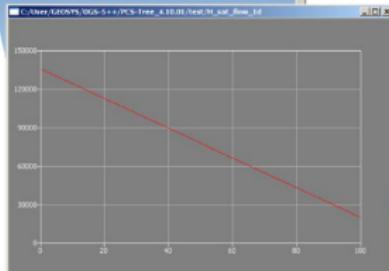
Datum	V	Thema	.
17.04.2020	00	Einführung in GoToMeeting (Web-Conferencing)	
17.04.2020	01	Einführung in die Lehrveranstaltung	
24.04.2020	02	Grundlagen: Kontinuumsmechanik	
08.05.2020	03	Grundlagen: Hydromechanik	
15.05.2020	04	Grundlagen: Partielle Partialgleichungen	
22.05.2020	05	Installation: Python, Qt C++	
29.05.2020	06	Grundlagen: Näherungsverfahren	
05.06.2020	07	Übungen: Übersicht und Werkzeuge	
12.06.2020	08	Numerik: Finite-Differenzen-Methode (explizit)	
19.06.2020	09	Numerik: Finite-Differenzen-Methode (implizit)	
26.06.2020	10	Anwendung: Gerinnehydraulik (Theorie)	
03.07.2020	11	Anwendung: Gerinnehydraulik (Übung)	
10.07.2020	12	Anwendung: Grundwassermodellierung (datenbasierte Methoden)	
17.07.2020	13	Beleg: Besprechung zur Vorbereitung	

Konzept

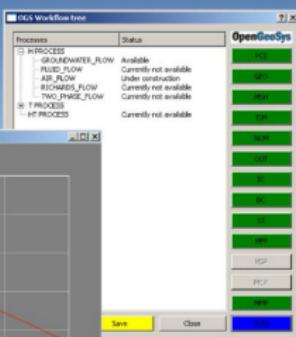
$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla \psi$$



Numerische
Methoden

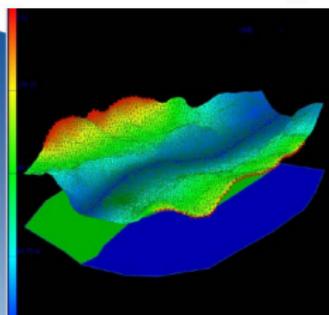


Basics
Mechanik



Prozessverständnis

Anwendung



Programmierung
Visual C++

Übung

- ▶ Python ...
-

Vorlesung

- ▶ Grundlagen der Finite Differenzen Methode
- ▶ Approximation methods
- ▶ Finite difference method – FDM (Ch. 3)
- ▶ Taylor series expansion
- ▶ Derivatives
- ▶ Diffusion equation
- ▶ (Finite element method – FEM ⇒ Hydrosystemanalyse)

Python, die Zweite (Path)

<https://www.python.org/downloads/>

The screenshot shows the Python Software Foundation's download page. At the top, there's a navigation bar with links for Python, PSF, Docs, PyPI, Jobs, and Community. Below the navigation is the Python logo. A search bar with a magnifying glass icon and a 'GO' button is positioned next to it. A 'Socialize' button is also visible. The main content area features a large blue banner with the text "Download the latest version for Windows". It includes a yellow "Download Python 3.7.3" button and links for other operating systems like Linux/UNIX, Mac OS X, and Other. There's also information about pre-releases and Docker images. A graphic of two packages descending from the sky with parachutes is displayed. At the bottom, there's a yellow call-to-action button that says "Contribute to the PSF by Purchasing a PyCharm License. All proceeds benefit the PSF." with a "Donate Now" button.

Looking for a specific release?

Python releases by version number:

Release version

Release date

Click for more

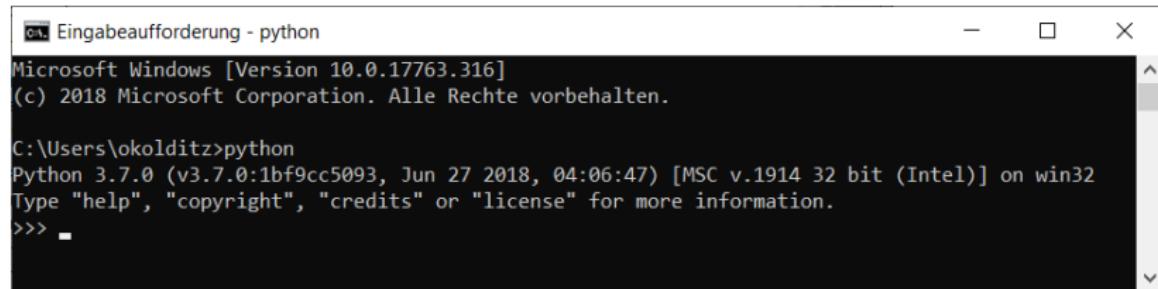
Python, die Zweite (Path)



Python, die Zweite (Path)

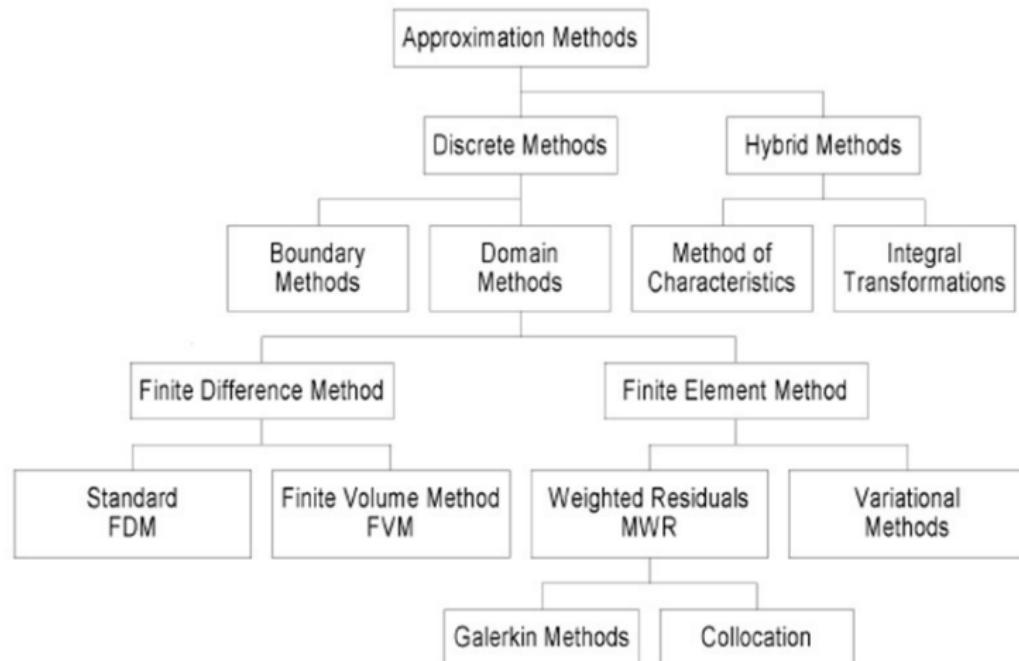
Zwei Optionen:

- 1 Python deinstallieren, neu installieren und "Add Python to PATH"
- 2 "PATH" nachträglich ergänzen (unterschiedlich für verschiedene Windows-Versionen), am besten googeln



```
Eingabeaufforderung - python
Microsoft Windows [Version 10.0.17763.316]
(c) 2018 Microsoft Corporation. Alle Rechte vorbehalten.

C:\Users\okolditz>python
Python 3.7.0 (v3.7.0:1bf9cc5093, Jun 27 2018, 04:06:47) [MSC v.1914 32 bit (Intel)] on win32
Type "help", "copyright", "credits" or "license" for more information.
>>> -
```



FDM Anwendungen - MODFLOW

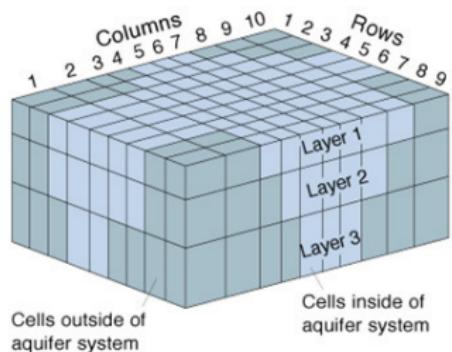


Figure 2. Example of model grid for simulating three-dimensional ground-water flow.

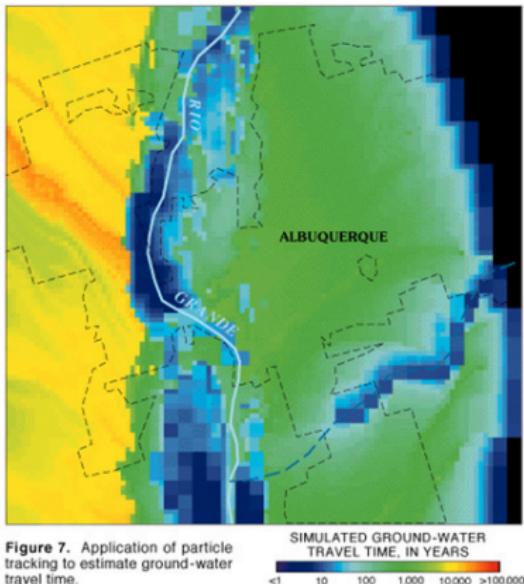
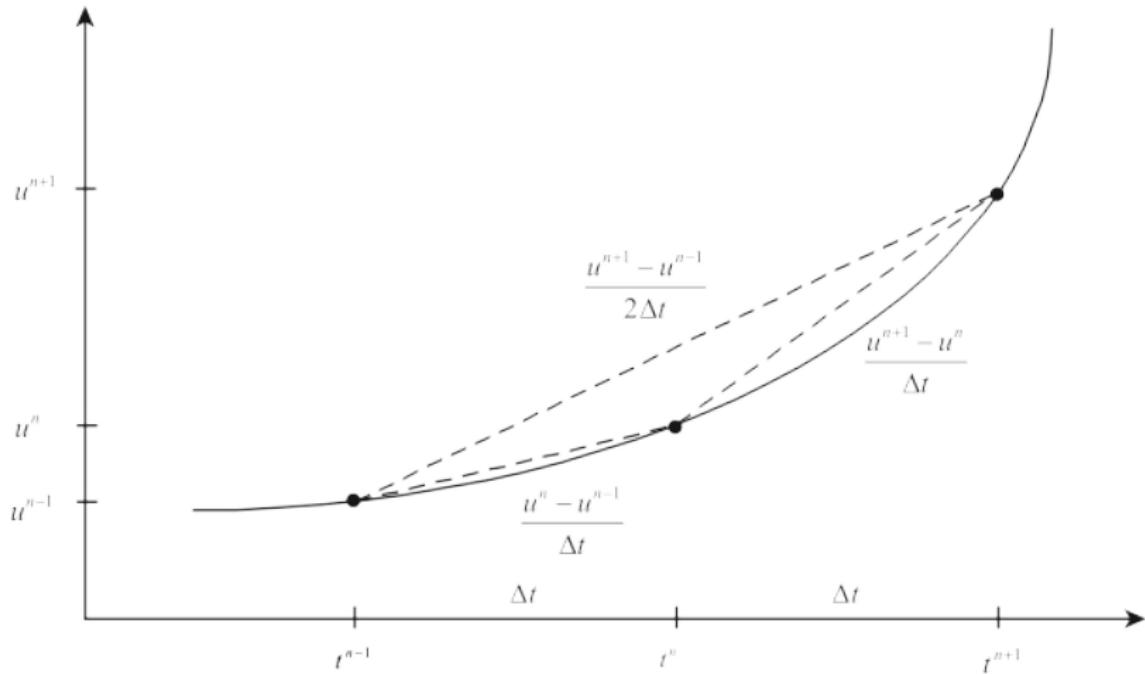
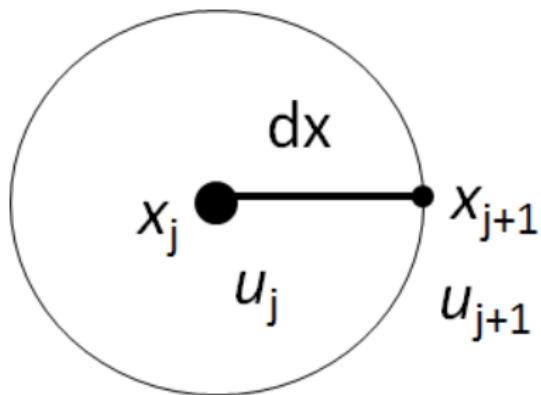


Figure 7. Application of particle tracking to estimate ground-water travel time.

<http://water.usgs.gov/pubs/FS/FS-121-97/images/fig7.gif>

Ableitungen





in time

$$u_j^{n+1} = \sum_{m=0}^{\infty} \frac{\Delta t^m}{m!} \left[\frac{\partial^m u}{\partial t^m} \right]_j^n \quad (1)$$

$$\Delta t = t^{n+1} - t^n$$

in space

$$u_{j+1}^n = \sum_{m=0}^{\infty} \frac{\Delta x^m}{m!} \left[\frac{\partial^m u}{\partial x^m} \right]_j^n \quad (2)$$

$$\Delta x = x_{j+1} - x_j$$

$$u_j^{n+1} = u_j^n + \Delta t \left[\frac{\partial u}{\partial t} \right]_j^n + \frac{\Delta t^2}{2} \left[\frac{\partial^2 u}{\partial t^2} \right]_j^n + O(\Delta t^3) \quad (3)$$

$$u_{j+1}^n = u_j^n + \Delta x \left[\frac{\partial u}{\partial x} \right]_j^n + \frac{\Delta x^2}{2} \left[\frac{\partial^2 u}{\partial x^2} \right]_j^n + O(\Delta x^3) \quad (4)$$

1. Ableitung

$$\left[\frac{\partial u}{\partial t} \right]_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\Delta t}{2} \left[\frac{\partial^2 u}{\partial t^2} \right]_j^n + o(\Delta t^2) \quad (5)$$

$$\left[\frac{\partial u}{\partial x} \right]_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} - \frac{\Delta x}{2} \left[\frac{\partial^2 u}{\partial x^2} \right]_j^n + o(\Delta x^2) \quad (6)$$

Differenzen-Schemata

Forward difference approximation

$$\left[\frac{\partial u}{\partial x} \right]_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} + O(\Delta x) \quad (7)$$

Backward difference approximation

$$\left[\frac{\partial u}{\partial x} \right]_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x} + O(\Delta x) \quad (8)$$

Central difference approximation

$$\left[\frac{\partial u}{\partial x} \right]_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O(\Delta x^2) \quad (9)$$

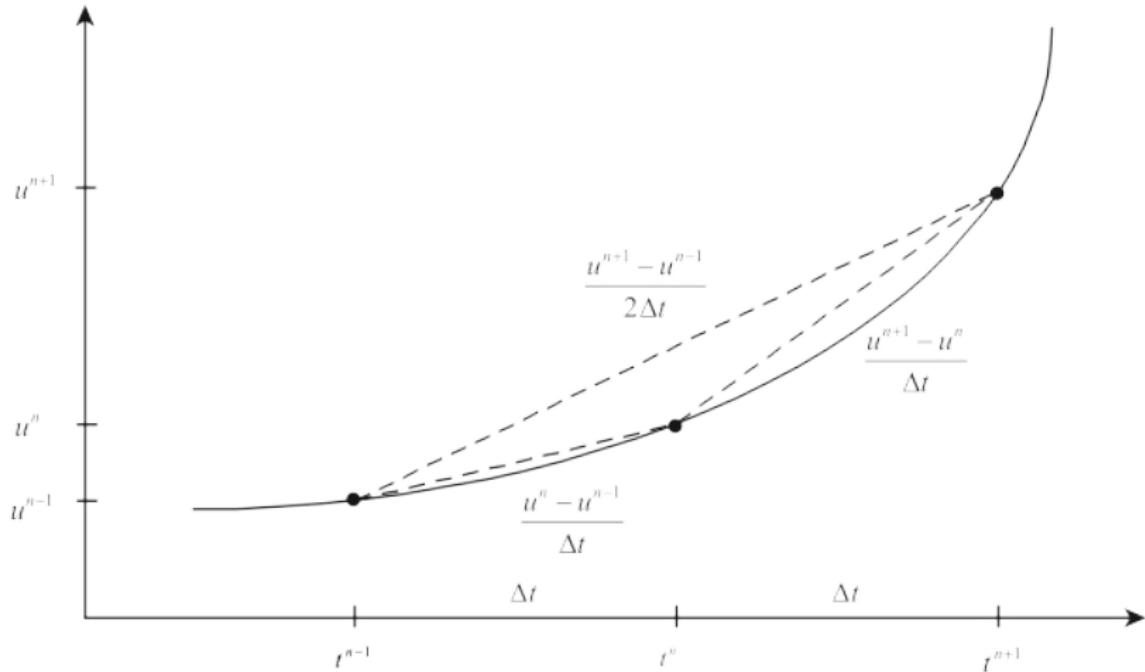
Zentrale Differenzen

$$\begin{aligned} u_{j+1}^n &= u_j^n + \Delta x \left[\frac{\partial u}{\partial x} \right]_j^n + \frac{\Delta x^2}{2} \left[\frac{\partial^2 u}{\partial x^2} \right]_j^n + O(\Delta x^3) \\ u_{j-1}^n &= u_j^n - \Delta x \left[\frac{\partial u}{\partial x} \right]_j^n + \frac{\Delta x^2}{2} \left[\frac{\partial^2 u}{\partial x^2} \right]_j^n - O(\Delta x^3) \quad (10) \end{aligned}$$

Central difference approximation

$$\left[\frac{\partial u}{\partial x} \right]_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O(\Delta x^2) \quad (11)$$

Ableitungen

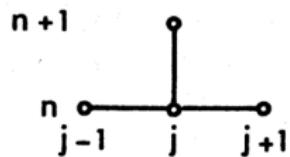


2. Ableitung

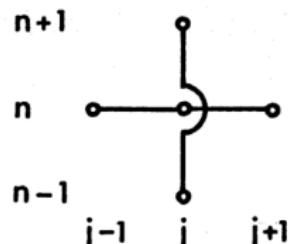
$$\begin{aligned}\left[\frac{\partial^2 u}{\partial x^2} \right]_j^n &\approx \frac{1}{\Delta x} \left(\left[\frac{\partial u}{\partial x} \right]_{j+1}^n - \left[\frac{\partial u}{\partial x} \right]_j^n \right) \\ &\approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\end{aligned}\tag{12}$$

$$\left[\frac{\partial^2 u}{\partial x^2} \right]_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{\Delta x^2}{12} \left[\frac{\partial^4 u}{\partial x^4} \right]_j^n + \dots\tag{13}$$

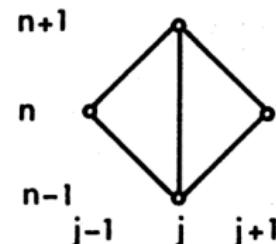
Übersicht Differenzenverfahren



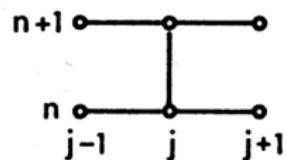
FTCS



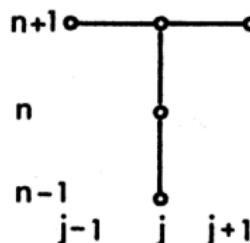
Richardson



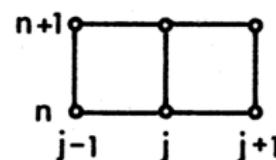
DuFort-Frankel



Crank-Nicolson



3LF1



Linear F.E.M./
Crank-Nicolson

Diffusionsgleichung

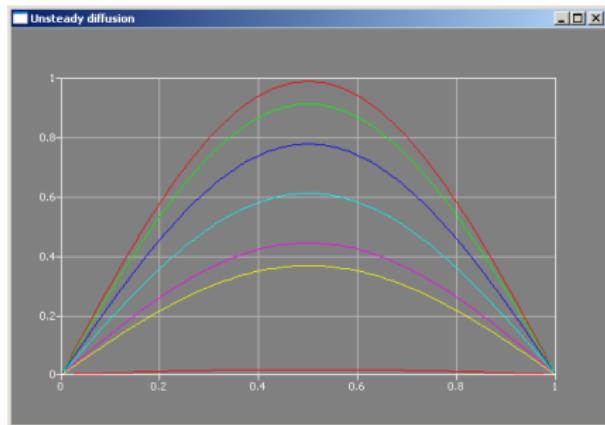
$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (14)$$

- ▶ Diffusion equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (15)$$

- ▶ Analytical solution

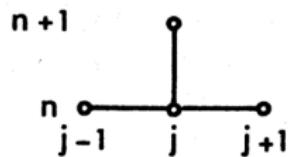
$$u = \sin(\pi x) e^{-\alpha t^2} \quad (16)$$



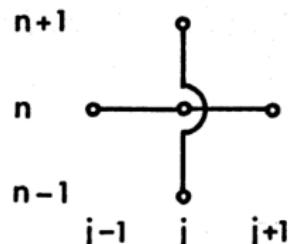
⇒ Übung

- ▶ K: validity

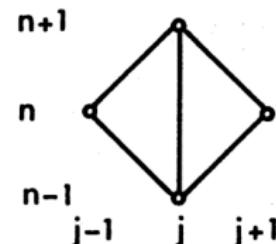
Übersicht Differenzenverfahren



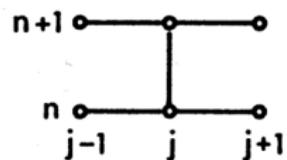
FTCS



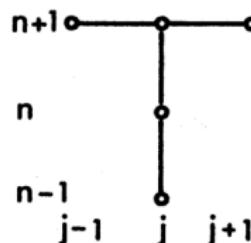
Richardson



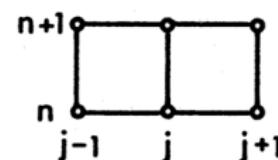
DuFort-Frankel



Crank-Nicolson



3LF1



Linear F.E.M./
Crank-Nicolson

- ▶ PDE for diffusion processes

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (17)$$

- ▶ forward time / centered space

$$\left[\frac{\partial u}{\partial t} \right]_j^n \approx \frac{u_j^{n+1} - u_j^n}{\Delta t} \quad \left[\frac{\partial^2 u}{\partial x^2} \right]_j^n \approx \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} \quad (18)$$

- ▶ substitute

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} = 0 \quad (19)$$

- ▶ FTCS scheme for diffusion equations

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2} (u_{j-1}^n - 2u_j^n + u_{j+1}^n), \quad Ne = \frac{\alpha \Delta t}{\Delta x^2} \quad (20)$$

Analysis of approximation schemes consists of three steps:

- ▶ Develop the **algebraic scheme**,
- ▶ Check **consistency** of the algebraic approximate equation,
- ▶ Investigate **stability** behavior of the scheme.

Eigenschaften numerischer Verfahren

Analysis of approximation schemes consists of three steps:

- ▶ Develop the **algebraic scheme**,

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2} (u_{j-1}^n - 2u_j^n + u_{j+1}^n) \quad (21)$$

- ▶ Check **consistency** of the algebraic approximate equation,

$$\lim_{\Delta t, \Delta x \rightarrow 0} |\hat{L}(u_j^n) - L(u[t_n, x_j])| = 0 \quad (22)$$

- ▶ Investigate **stability** behavior of the scheme.

$$Ne = \frac{\alpha \Delta t}{\Delta x^2} \leq 1/2 \quad (23)$$

Lösung des FTCS Schemas

Algebraische Schema

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2} (u_{j-1}^n - 2u_j^n + u_{j+1}^n) \quad (24)$$

Resultierendes Gleichungssystem

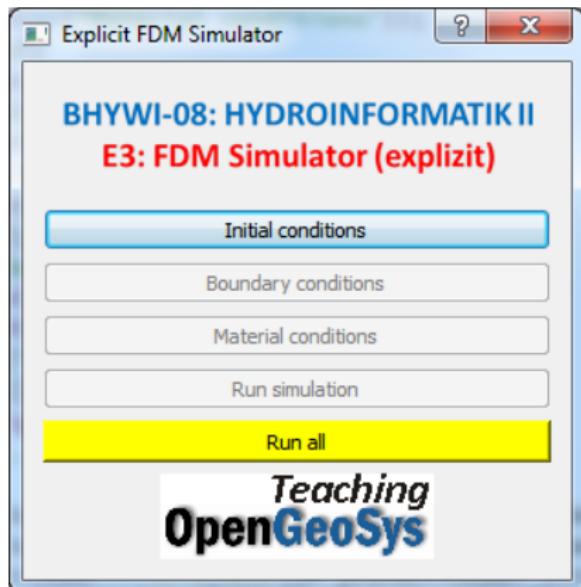
$$\mathbf{u}^{n+1} = \mathbf{A}\mathbf{u}^n \quad , \quad n = 0, 1, 2, \dots \quad (25)$$

$$\mathbf{A} = \begin{bmatrix} 1 - 2\frac{\alpha \Delta t}{\Delta x^2} & \frac{\alpha \Delta t}{\Delta x^2} & & \\ \frac{\alpha \Delta t}{\Delta x^2} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{\alpha \Delta t}{\Delta x^2} \\ & & & \frac{\alpha \Delta t}{\Delta x^2} & 1 - 2\frac{\alpha \Delta t}{\Delta x^2} \end{bmatrix}, \quad \mathbf{u}^n = \begin{bmatrix} u_2^n \\ u_3^n \\ \vdots \\ u_{np-2}^n \\ u_{np-1}^n \end{bmatrix}$$

Übung BHYWI-08-03-E

FDM: Explizit

Programm-Dialog

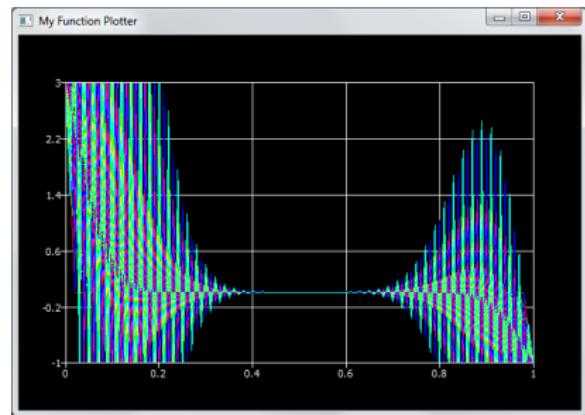
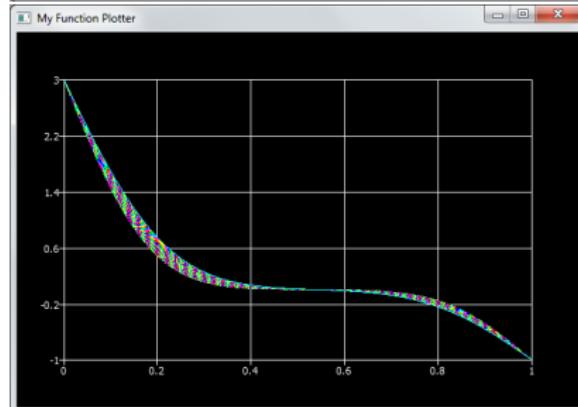
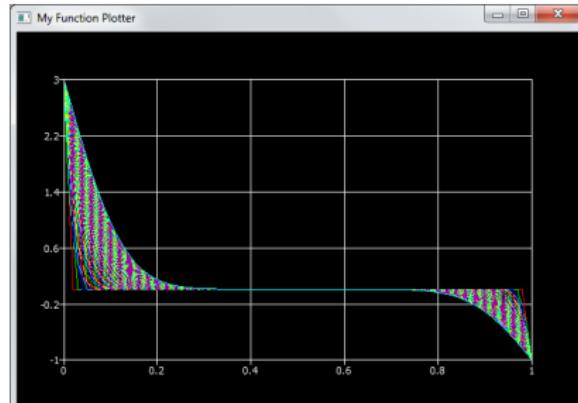


Dialog-Klasse: Konstruktor
Dialog::Dialog

- 1 Elemente
- 2 Connects
- 3 Layout
- 4 Datenstrukturen
(Speicherreservierung)

FDM: Explizit

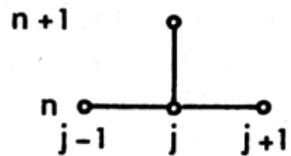
Ergebnisse



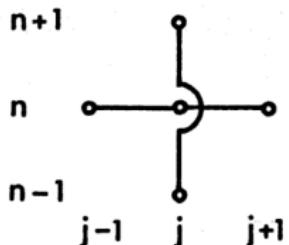
$$Ne = \frac{\alpha \Delta t}{\Delta x^2} \leq 0.5 \quad (26)$$

How sensitive ?

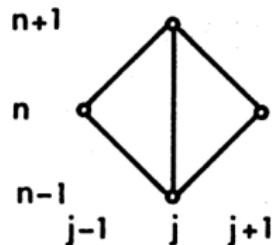
Explizite und implizite Differenzenverfahren



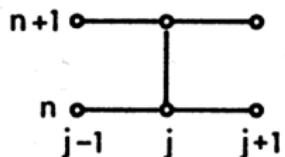
FTCS



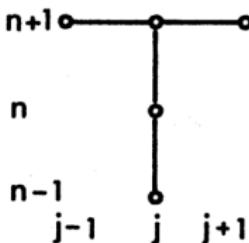
Richardson



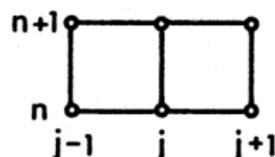
DuFort-Frankel



Crank-Nicolson



3LFI



Linear F.E.M./
Crank-Nicolson

Implizites Differenzenverfahren: Next Lecture

Algebraische Schema:

$$\left[\frac{\partial^2 u}{\partial x^2} \right]_j^{n+1} \approx \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} \quad (27)$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} = 0 \quad (28)$$

$$\frac{\alpha \Delta t}{\Delta x^2} (-u_{j-1}^{n+1} + 2u_j^{n+1} - u_{j+1}^{n+1}) + u_j^{n+1} = u_j^n \quad (29)$$